20 years trekking on the LS path: Talk of Anu Rama at Seshadri 80th birthday conference 23-27 January 2012, CHT, Chennai.

Before the trek

1987  Weyl, The classical groups
1988  Macdonald, Symmetric Functions and Hall polys
1990  Littelmann, A generalization of the LR rule
Lakshmibai - Seshadri, Geometry of G/P/I.
1992  Lakshmibai - Seshadri, Standard Monomial Theory
Hyderabad conference volume
Lakshmibai - Littelmann email
"I proved your conjecture"

\[ \chi_{s}(H^0(G/B, L^1)) = \sum_{w \in W_0} \det(w) e^{w(\alpha + \gamma)} \]
\[ = \sum_{w \in W_0} e^{\rho T \left( 1 - \frac{1}{e^w} \right)} \]
\[ \text{p \in B(\Lambda)} \]

Views from the LS path

Crystals, Affine Hecke algebra
Schubert calculus, Loop groups

... the horizon.
Data

\( G = \) complex reductive algebraic group

\( B = \) Borel subgroup

\( T = \) maximal torus

The Weyl group is \( W_0 = N(T)/T \) acts on

\[ \mathfrak{h}^* = \text{Hom}(T, \mathbb{C}^*) \text{ and } \mathfrak{b}^* = \text{Hom}(C^*, T) \]

\( C \) is a fundamental chamber for \( W_0 \)-action on \( \mathfrak{h}^* = \mathbb{R}^\vee \otimes \mathfrak{h}^* \)

\( W_0 \) is generated by \( s_1, s_2, \ldots, s_n \) reflections in the walls \( \mathfrak{h}_i, \ldots, \mathfrak{h}_n \) of \( C \).

Type \( S_3 \)

\[ p = (a_1, a_2, a_3, a_4) \]

initial direction = \( \alpha(p) \)

\( \varphi(p) = \) final direction
$G/B$ is the flag variety and

$$G = \bigcup_{w \in \mathfrak{g}} BwB$$

and $x_w = \overline{BwB}$ in $G/B$

are the Schubert varieties.

**Crystals** For $i = 1, 2, \ldots, n$ define $\tilde{F}_i: \text{paths} \to \text{paths/0}$

For $\lambda \in \mathfrak{g}^{\ast}_C (C - p)$ let $p_\lambda: [0, 1] \to \mathbb{R}^n$ with $p_\lambda(1) = \lambda$ and $p_\lambda([0, 1]) \subseteq C - p$. Let

$$B(\lambda) = \{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_m} \lambda | m \in \mathbb{Z}_{\geq 0}, 1 \leq i_1, \ldots, i_m \leq n \}$$

let $w = s_{i_1} \cdots s_{i_j}$ be minimal length. Then

$$\text{char}_C(\lambda, x_w) = \tilde{T}_{i_1} \cdots \tilde{T}_{i_j} e^\lambda = \sum_{\rho \leq B(\lambda) \subseteq w} e^\rho$$

where

$$e^\lambda = e^{G \times B \cdot \lambda}, \quad \tilde{T}_i = e^p \frac{1}{1 - e^{-\varepsilon_i} (1 - s_i) e^p}$$

and

$$B(\lambda)_{x_w} = \{ \rho \in B(\lambda) | \text{in}(\rho) \subseteq w \}$$
The operators

\[ T_i : \text{Rep}(T) \rightarrow \text{Rep}(T) \quad \text{and} \quad X^\mu : \text{Rep}(T) \rightarrow \text{Rep}(T) \]

provide a representation of the nil affine Hecke algebra \( H_{10} \) has generators \( T_i \) and \( X^\mu \)
relations

\[ T_i^2 = T_i \quad \text{and} \quad T_i T_j T_i \cdots = T_j T_i T_i \cdots \]

\[ T_i X^\mu = X^\mu T_i + \frac{X^\mu X_i^\mu}{1 - X^\mu X_i^\mu} \]

Theorem: Let \( \lambda \in \mathfrak{h}^* \backslash \mathfrak{h}/(-\rho) \) and \( \nu \in W_0 \).

(a) In \( H_{10} \),

\[ T_{w_1} X^\mu = \sum_{\gamma \in B(\mathfrak{h})_{w_1}} x_{\text{end}(\gamma)} T_{w_2} \]

(b) In \( X_+ (G/\mathcal{B}) \),

\[ [L_{\lambda}, [L_{\mu}, L_{\nu}] = \sum_{\gamma \in B(\mathfrak{h})_{w}} e_{\text{end}(\gamma)} [L_{\gamma(\lambda)}] \]

(c) The \( T \)-equivariant sheaf on \( G/\mathcal{B} \), \( F = L_{\lambda} \otimes L_{\mu} \)

has

\[ F \supset F(1) \supset F(2) \supset \cdots \]

so that the quotients are \( e_{\text{end}(\gamma)} \otimes L_{\gamma(\rho)} \).
Loop groups

\[ G = G(\mathcal{L}(t)) \]

\[ L^1 = G(\mathcal{D}(t)) \xrightarrow{t \mapsto 0 \oplus t} G(\mathcal{L}(t)) \]

\[ L^1 \]

\[ I = \mathbb{H}^1 / B \rightarrow B \]

\( G \) is presented by generators \( x_\alpha(t), x_\alpha(t)^{-1} \) and \( h_\mu(q) \)

\( x_\alpha \in \mathbb{R}^+, \ \nu \in \mathbb{R}^+, \ f \in \mathcal{L}(t), \ \rho \in \mathcal{L}(t)^* \) with

Steinberg-Tits relations

Let \( U = \langle x_\alpha(t) \mid x_\alpha \in \mathbb{R}^+, f \in \mathcal{L}(t) \rangle = \{ 1, i \} \)

The affine Weyl group is

\[ W = W_0 \times \mathbb{Z} \]

where \( t_\alpha = h_\mu(t) \). Then

\[ G = L^1 I W I, \quad G = U U V I \]

\[ G = L^1 K t_\alpha K \quad G = L^1 U^{-1} t_\alpha K \]

The Mirkevič-Vilonen intersections are

\[ I W I \cap U^{-1} V I \quad \text{and} \quad K t_\alpha K \cap U^{-1} t_\alpha K. \]
Theorem

\{ \text{labeled positively folded walk} \} \overset{1-1}{\longleftrightarrow} \text{I}_W \cap \text{I}_V

First note

\[ W \leadsto \text{fallover} \]

\[ x_k x_{k+8}(c) = x_k (c t^k) \]

Fix \( \bar{w} = \bar{s}_1 \bar{s}_2 \cdots \bar{s}_l \) minimal length to \( w \).

A step \( \bar{p}_j \) is

\[
\begin{array}{ccc}
\bar{s}_j & \bar{s}_j & \bar{s}_j \\
\rightarrow & \rightarrow & \rightarrow \\
\text{d}_j \in \mathbb{Z} & \text{d}_j \in \mathbb{Z} & \text{d}_j \in \mathbb{Z}
\end{array}
\]

where the periodic orientation is given by

(a) hyperplanes through 0 have \( \bar{d} \) an positive side

(b) parallel hyperplanes have parallel orientation

\[ \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_l) \quad \overset{\text{I}_W}{\longrightarrow} \quad \bar{x}_1(\text{d}_1) \bar{x}_2(\text{d}_2) \cdots \bar{x}_l(\text{d}_l) \text{I}_V \]
The horizon

The diagram includes various mathematical notations and symbols, which appear to be related to algebraic structures, such as groups, rings, and modules. The diagram seems to represent relationships between different algebraic concepts, possibly involving the study of Lie algebras, algebraic geometry, or related fields.

The text around the diagram includes terms like "H" and "Z_p = H V_p" which might be part of a mathematical expression or theorem. The diagram itself is complex and includes various arrows and connections indicating flow or relationships between different concepts.