(3.1.3) Proposition. If $f: K \to F$ is a field homomorphism then $f$ is injective.

Proof.
To show: $f: K \to F$ is injective.
Assume $f: K \to F$ is a field homomorphism.
To show: If $x_1, x_2 \in K$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$.
Assume $x_1, x_2 \in K$ and $f(x_1) = f(x_2)$.
To show: $x_1 = x_2$.
Proof by contradiction: Assume $x_1 \neq x_2$.
Let $0_K$ and $0_F$ be the additive identities in $K$ and $F$ respectively.
Let $1_K$ and $1_F$ be the multiplicative identities in $K$ and $F$ respectively.
Then $f(x_1) - f(x_2) = 0_F$ and $x_1 - x_2 \neq 0_K$.
Let $y = (x_1 - x_2)^{-1}$, which exists by property h) in the definition of a field.
Then, since $f: K \to F$ is a homomorphism and $f(x_1) - f(x_2) = 0_F$,
\[
1_F = f(1_K) = f((x_1 - x_2)y) \\
= f(x_1 - x_2)f(y) \\
= (f(x_1) - f(x_2))f(y) \\
= 0_F \cdot f(y) \\
= 0_F.
\]
This is a contradiction to property g) in the definition of a field.
So $x_1 = x_2$.
So $f: K \to F$ is injective. □
§2P. Vector Spaces

(3.2.4) Proposition. Let \( V \) be a vector space over a field \( F \) and let \( W \) be a subgroup of \( V \). Then the cosets of \( W \) in \( V \) partition \( V \).

Proof.
To show:  a) If \( v \in V \) then \( v \in v' + W \) for some \( v' \in V \).

b) If \( (v_1 + W) \cap (v_2 + W) \neq \emptyset \) then \( v_1 + W = v_2 + W \).

a) Let \( v \in V \).
Then, since \( 0 \in W \), \( v = v + 0 \in v + W \).
So \( v \in v + W \).

b) Assume \( (v_1 + W) \cap (v_2 + W) \neq \emptyset \).
To show: ba) \( v_1 + W \subseteq v_2 + W \).
bb) \( v_2 + W \subseteq v_1 + W \).
Let \( a \in (v_1 + W) \cap (v_2 + W) \).
Suppose \( a = v_1 + w_1 \) and \( a = v_2 + w_2 \) where \( w_1, w_2 \in W \).

Then
\[
\begin{align*}
v_1 &= v_1 + w_1 - w_1 = a - w_1 = v_2 + w_2 - w_1 \quad \text{and} \\
v_2 &= v_2 + w_2 - w_2 = a - w_2 = v_1 + w_1 - w_2.
\end{align*}
\]

ba) Let \( v \in v_1 + W \).
Then \( v = v_1 + w \) for some \( w \in W \).

Then
\[
v = v_1 + w = v_2 + w_2 - w_1 + w \in v_2 + W,
\]
since \( w_2 - w_1 + w \in W \).
So \( v_1 + W \subseteq v_2 + W \).

bb) Let \( v \in v_2 + W \).
Then \( v = v_2 + w \) for some \( w \in W \).

Then
\[
v = v_2 + w = v_1 + w_1 - w_2 + w \in v_1 + W,
\]
since \( w_1 - w_2 + w \in W \).
So \( v_2 + W \subseteq v_1 + W \).

So the cosets of \( W \) in \( V \) partition \( V \). \( \square \)

(3.2.5) Theorem. Let \( W \) be a subgroup of a vector space \( V \) over a field \( F \). Then \( W \) is a subspace of \( V \) if and only if \( V/W \) with operations given by
\[
(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W, \quad \text{and} \quad c(v + W) = cv + W,
\]
is a vector space over \( F \).

Proof.
\( \Rightarrow \): Assume \( W \) is a subspace of \( V \).
To show: a) \( (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \) is a well defined operation on \( V/W \).

b) The operation given by \( c(v + W) = cv + W \) is well defined.

c) \((v_1 + W) + (v_2 + W)) + (v_3 + W) = (v_1 + W) + ((v_2 + W) + (v_3 + W))\)
for all \( v_1 + W, v_2 + W, v_3 + W \in V/W \).

d) \((v_1 + W) + (v_2 + W) = (v_2 + W) + (v_1 + W)\) for all \( v_1 + W, v_2 + W \in V/W \).
e) $0 + W = W$ is the zero in $V/W$.
f) $-v + W$ is the additive inverse of $v + W$.
g) If $c_1, c_2 \in F$ and $v + W \in V/W$, then $c_1 (c_2 (v + W)) = (c_1 c_2) (v + W)$.
h) If $v + W \in V/W$ then $1 (v + W) = v + W$.
i) If $c \in F$ and $v_1 + W, v_2 + W \in V/W$, then $c ((v_1 + W) + (v_2 + W)) = c (v_1 + W) + c (v_2 + W)$.
j) If $c_1, c_2 \in F$ and $v + W \in V/W$, then $(c_1 + c_2) (v + W) = c_1 (v + W) + c_2 (v + W)$.

a) We want the operation on $V/W$ given by

$$V/W \times V/W \rightarrow V/W$$

$$(v_1 + W, v_2 + W) \mapsto (v_1 + v_2) + W$$

to be well defined.

Let $(v_1 + W, v_2 + W), (v_3 + W, v_4 + W) \in V/W \times V/W$ such that

$$(v_1 + W, v_2 + W) = (v_3 + W, v_4 + W).$$

Then $v_1 + W = v_3 + W$ and $v_2 + W = v_4 + W$.

To show: $(v_1 + v_2) + W = (v_3 + v_4) + W$.

So we must show: aa) $(v_1 + v_2) + W \subseteq (v_3 + v_4) + W$.

ab) $(v_3 + v_4) + W \subseteq (v_1 + v_2) + W$.

aa) We know $v_1 = v_1 + 0 \in v_3 + W$ since $v_1 + W = v_3 + W$.

So $v_1 = v_3 + w_1$ for some $w_1 \in W$.

Similarly $v_2 = v_4 + w_2$ for some $w_2 \in W$.

Let $t \in (v_1 + v_2) + W$.

Then $t = v_1 + v_2 + w$ for some $w \in W$.

So

$$t = v_1 + v_2 + w$$

$$= v_3 + w_1 + v_4 + w_2 + w$$

$$= v_3 + v_4 + w_1 + w_2 + w,$$

since addition is commutative.

So $t = (v_3 + v_4) + (w_1 + w_2 + w) \in v_3 + v_4 + W$.

So $(v_1 + v_2) + W \subseteq (v_3 + v_4) + W$.

ab) Since $v_1 + W = v_3 + W$, we know $v_1 + w_1 = v_3$ for some $w_1 \in W$.

Since $v_2 + W = v_4 + W$, we know $v_2 + w_2 = v_4$ for some $w_2 \in W$.

Let $t \in (v_3 + v_4) + W$.

Then $t = v_3 + v_4 + w$ for some $w \in W$.

So

$$t = v_3 + v_4 + w$$

$$= v_1 + w_1 + v_2 + w_2 + w$$

$$= v_1 + v_2 + w_1 + w_2 + w,$$

since addition is commutative.

So $t = (v_1 + v_2) + (w_1 + w_2 + w) \in (v_1 + v_2) + W$.

So $(v_3 + v_4) + W \subseteq (v_1 + v_2) + W$.

So $(v_1 + v_2) + W = (v_3 + v_4) + W$.

So the operation given by $(v_1 + W) + (v_3 + W) = (v_1 + v_3) + W$ is a well defined operation on $V/W$.

b) We want the operation given by

$$F \times V/W \rightarrow V/W$$

$$(c, v + W) \rightarrow cv + W$$

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to be well defined.

Let \((c_1, v_1 + W), (c_2, v_2 + W) \in (F \times V/W)\) such that \((c_1, v_1 + W) = (c_2, v_2 + W)\).

Then \(c_1 = c_2\) and \(v_1 + W = v_2 + W\).

To show: \(c_1 v_1 + W = c_2 v_2 + W\).

To show: ba) \(c_1 v_1 + W \subseteq c_2 v_2 + W\).

bb) \(c_2 v_2 + W \subseteq c_1 v_1 + W\).

ba) Since \(v_1 + W = v_2 + W\), we know \(v_1 = v_2 + w_1\) for some \(w_1 \in W\).

Let \(t \in c_1 v_1 + W\).

Then \(t = c_1 v_1 + w\) for some \(w \in W\). So

\[
t = c_1 v_1 + w = c_2 (v_2 + w_1) + w = c_2 v_2 + c_2 w_1 + w,
\]

since \(c_1 = c_2\).

Since \(W\) is a subspace, \(c_2 w_1 \in W\), and \(c_2 w_1 + w \in W\).

So \(t = c_2 v_2 + c_2 w_1 + w \in c_2 v_2 + W\).

So \(c_1 v_1 + W \subseteq c_2 v_2 + W\).

bb) Since \(v_1 + W = v_2 + W\), we know \(v_2 = v_1 + w_2\) for some \(w_2 \in W\).

Let \(t \in c_2 v_2 + W\).

Then \(t = c_2 v_2 + w\) for some \(w \in W\). So

\[
t = c_2 v_2 + w = c_1 (v_1 + w_2) + w = c_1 v_1 + c_1 w_2 + w,
\]

since \(c_2 = c_1\).

Since \(W\) is a subspace, \(c_1 w_2 \in W\), and \(c_1 w_2 + w \in W\).

So \(t = c_1 v_1 + c_1 w_2 + w \in c_1 v_1 + W\).

So \(c_2 v_2 + W \subseteq c_1 v_1 + W\).

So \(c_1 v_1 + W = c_2 v_2 + W\).

So the operation is well defined.

c) By the associativity of addition in \(V\) and the definition of the operation in \(V/W\),

\[
((v_1 + W) + (v_2 + W)) + (v_3 + W) = ((v_1 + v_2) + W) + (v_3 + W)
\]

\[
= ((v_1 + v_2) + v_3) + W
\]

\[
= (v_1 + (v_2 + v_3)) + W
\]

\[
= (v_1 + W) + ((v_2 + v_3) + W)
\]

\[
= (v_1 + W) + ((v_2 + W) + (v_3 + W))
\]

for all \(v_1 + W, v_2 + W, v_3 + W \in V/W\).

d) By the commutativity of addition in \(V\) and the definition of the operation in \(V/W\),

\[
(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W
\]

\[
= (v_2 + v_1) + W
\]

\[
= (v_2 + W) + (v_1 + W).
\]

for all \(v_1 + W, v_2 + W \in V/W\).

e) The coset \(W = 0 + W\) is the zero in \(V/W\) since
\[
W + (v + W) = (0 + v) + W \\
= v + W \\
= (v + 0) + W \\
= (v + W) + W
\]

for all \( v + W \in V/W \).

f) Given any coset \( v + W \), its additive inverse is \((-v) + W\) since

\[
(v + W) + (-v + W) = v + (-v) + W \\
= 0 + W \\
= W \\
= (-v + v) + W \\
= (-v + W) + v + W
\]

for all \( v + W \in V/W \).

g) Assume \( c_1, c_2 \in F \) and \( v + W \in V/W \). Then, by definition of the operation,

\[
c_1 (c_2 (v + W)) = c_1 (c_2 v + W) \\
= c_1 (c_2 v) + W \\
= (c_1 c_2) v + W \\
= (c_1 c_2) (v + W).
\]

h) Assume \( v + W \in V/W \). Then, by definition of the operation,

\[
1(v + W) = (1v) + W \\
= v + W.
\]

i) Assume \( c \in F \) and \( v_1 + W, v_2 + W \in V/W \). Then

\[
c((v_1 + W) + (v_2 + W)) = c((v_1 + v_2) + W) \\
= c(v_1 + v_2) + W \\
= (cv_1 + cv_2) + W \\
= (cv_1 + W) + (cv_2 + W) \\
= c(v_1 + W) + c(v_2 + W).
\]

j) Assume \( c_1, c_2 \in F \) and \( v + W \in V/W \). Then

\[
(c_1 + c_2)(v + W) = ((c_1 + c_2)v) + W \\
= (c_1 v + c_2 v) + W \\
= (c_1 v + W) + (c_2 v + W) \\
= c_1 (v + W) + c_2 (v + W).
\]

So \( V/W \) is a vector space over \( F \).

\[\leftarrow:\text{Assume } W \text{ is a subgroup of } V \text{ and } V/W \text{ is a vector space over } F \text{ with action given by}\]

5
\[ c(v + W) = cv + W. \]

To show: \( W \) is a subspace of \( V \).

To show: If \( c \in F \) and \( w \in W \) then \( cw \in W \).

First we show: If \( w \in W \) then \( w + W = W \).

To show: a) \( w + W \subseteq W \).

b) \( W \subseteq w + W \).

a) Let \( k \in w + W \).

So \( k = w + w_1 \) for some \( w_1 \in W \).

Since \( W \) is a subgroup, \( w + w_1 \in W \).

So \( w + W \subseteq W \).

b) Let \( k \in W \).

Since \( k - w \in W \), \( k = w + (k - w) \in w + W \).

So \( W \subseteq w + W \).

Now assume \( c \in F \) and \( w \in W \).

Then, by definition of the operation on \( V/W \),

\[
cw + W = c(w + W) \\
= c(0 + W) \\
= c \cdot 0 + W \\
= 0 + W \\
= W.
\]

So \( cw = cw + 0 \in W \).

So \( W \) is a subspace of \( V \).

\[ \square \]

(3.2.8) Proposition. Let \( T: V \to W \) be a linear transformation. Let \( 0_V \) and \( 0_W \) be the zeros for \( V \) and \( W \) respectively. Then

a) \( T(0_V) = 0_W \).

b) For any \( v \in V \), \( T(-v) = -T(v) \).

Proof.

a) Add \( -T(0_V) \) to both sides of the following equation.

\[
T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V).
\]

b) Since \( T(v) + T(-v) = T(v + (-v)) = T(0_V) = 0_W \) and \( T(-v) + T(v) = T((-v) + v) + T(0_V) = 0_W \), then

\[
-T(v) = T(-v). \quad \square
\]

(3.2.10) Proposition. Let \( T: V \to W \) be a linear transformation. Then

a) \( \ker T \) is a subspace of \( V \).

b) \( \text{im} T \) is a subspace of \( W \).

Proof.

a) By condition a) in the definition of linear transformation, \( T \) is a group homomorphism.

By Proposition 1.1.13 a), \( \ker T \) is a subgroup of \( V \).

To show: If \( c \in F \) and \( k \in \ker T \) then \( ck \in \ker T \).

Assume \( c \in F \) and \( k \in \ker T \).

Then, by the definition of linear transformation,

\[
T(ck) = cT(k) = c \cdot 0 = 0.
\]

So \( ck \in \ker T \).
So \( \ker T \) is a subspace of \( V \).

b) By condition a) in the definition of linear transformation, \( T \) is a group homomorphism.

By Proposition 1.1.13 b), \( \text{im } T \) is a subgroup of \( W \).

To show: If \( c \in F \) and \( a \in \text{im } T \) then \( ca \in \text{im } T \).

Assume \( c \in F \) and \( a \in \text{im } T \).

Then \( a = T(v) \) for some \( v \in V \).

By the definition of linear transformation,

\[
ca = cT(v) = T(cv).
\]

So \( ca \in \text{im } T \).
So \( \text{im } T \) is a subspace of \( W \). \( \square \)

(3.2.11) Proposition. Let \( T: V \to W \) be a linear transformation. Let \( 0_V \) be the zero in \( V \). Then

a) \( \ker T = (0_V) \) if and only if \( T \) is injective.

b) \( \text{im } T = W \) if and only if \( T \) is surjective.

Proof.

Let \( 0_V \) and \( 0_W \) be the zeros in \( V \) and \( W \) respectively.

a) \( \implies \): Assume \( \ker T = (0_V) \).

To show: If \( T(v_1) = T(v_2) \) then \( v_1 = v_2 \).

Assume \( T(v_1) = T(v_2) \).

Then, by the fact that \( T \) is a homomorphism,

\[
0_W = T(v_1) - T(v_2) = T(v_1 - v_2).
\]

So \( v_1 - v_2 \in \ker T \).
But \( \ker T = (0_V) \).
So \( v_1 - v_2 = 0_V \).
So \( v_1 = v_2 \).
So \( T \) is injective.

\( \iff \): Assume \( T \) is injective.

To show: \( (0_V) \subseteq \ker T \).

ab) \( \ker T \subseteq (0_V) \).

aa) Since \( T(0_V) = 0_W \), \( 0_V \in \ker T \).
So \( (0_V) \subseteq \ker T \).

ab) Let \( k \in \ker T \).
Then \( T(k) = 0_W \).
So \( T(k) = T(0_V) \).
Thus, since \( T \) is injective, \( k = 0_V \).
So \( \ker T \subseteq (0_V) \).
So \( \ker T = (0_V) \).

b) \( \implies \): Assume \( \text{im } T = W \).

To show: If \( w \in W \) then there exists \( v \in V \) such that \( T(v) = w \).

Assume \( w \in W \).
Then \( w \in \text{im } T \).
So there is some \( v \in V \) such that \( T(v) = w \).
So \( T \) is surjective.

\( \iff \): Assume \( T \) is surjective.

To show: \( \text{im } T \subseteq W \).

bb) \( W \subseteq \text{im } T \).

ba) Let \( x \in \text{im } T \).
Then \( x = T(v) \) for some \( v \in V \).
By the definition of $T$, $T(v) \in W$.
So $x \in W$.
So $\text{im} \ T \subseteq W$.

bb) Assume $x \in W$.
Since $T$ is surjective there is a $v$ such that $T(v) = x$.
So $x \in \text{im} \ T$.
So $W \subseteq \text{im} \ T$.

So $\text{im} \ T = W$. □

(3.2.12) Theorem.
a) Let $T : V \rightarrow W$ be a linear transformation and let $K = \ker T$. Define

$$
\hat{T} : V/\ker T \rightarrow W
\hat{T}(v + K) \rightarrow T(v).
$$

Then $\hat{T}$ is a well defined injective linear transformation.

b) Let $T : V \rightarrow W$ be a linear transformation and define

$$
T' : V \rightarrow \text{im} \ T
T(v) \rightarrow T(v).
$$

Then $T'$ is a well defined surjective linear transformation.

c) If $T : V \rightarrow W$ is a linear transformation, then

$$
V/\ker T \cong \text{im} \ T
$$

where the isomorphism is a vector space isomorphism.

Proof.
a) To show: aa) $\hat{T}$ is well defined.
ab) $\hat{T}$ is injective.
ac) $\hat{T}$ is a linear transformation.

aa) To show: aaa) If $v \in V$ then $\hat{T}(v + K) \in W$.

aab) If $v_1 + K = v_2 + K \in V/K$ then $\hat{T}(v_1 + K) = \hat{T}(v_2 + K)$.

aaa) Assume $v \in V$.
Then $\hat{T}(v + K) = T(v)$ and $T(v) \in W$, by the definition of $\hat{T}$ and $T$.

aab) Assume $v_1 + K = v_2 + K$.
Then $v_1 = v_2 + k$, for some $k \in K$.
To show: $T(v_1 + K) = T(v_2 + K)$, i.e.,
To show: $T(v_1) = T(v_2)$.
Since $K \in \ker T$, we have $T(k) = 0$ and so

$$
T(v_1) = T(v_2 + k) = T(v_2) + T(k) = T(v_2).
$$

So $\hat{T}(v_1 + K) = \hat{T}(v_2 + K)$.
So $\hat{T}$ is well defined.

ab) To show: If $\hat{T}(v_1 + K) = \hat{T}(v_2 + K)$ then $v_1 + K = v_2 + K$.

Assume $\hat{T}(v_1 + K) = \hat{T}(v_2 + K)$. Then $T(v_1) = T(v_2)$.
So $T(v_1) - T(v_2) = 0$.
So $T(v_1 - v_2) = 0$.
So $v_1 - v_2 \in \ker T$.
So $v_1 - v_2 = k$, for some $k \in \ker T$.
So $v_1 = v_2 + k$, for some $k \in \ker T$.
To show: aba) \( v_1 + K \subseteq v_2 + K \).

\[ \text{abb) } v_2 + K \subseteq v_1 + K. \]

aba) Let \( v \in v_1 + K \). Then \( v = v_1 + k_1 \), for some \( k_1 \in K \).
So \( v = v_2 + k + k_1 \in v_2 + K \), since \( k + k_1 \in K \).
So \( v_1 + K \subseteq v_2 + K \).

abb) Let \( v \in v_2 + K \). Then \( v = v_2 + k_2 \), for some \( k_2 \in K \).
So \( v = v_1 - k + k_2 \in v_1 + K \) since \( -k + k_2 \in K \).
So \( v_2 + K \subseteq v_1 + K \).

So \( v_1 + K = v_2 + K \).

So \( \hat{T} \) is injective.

ac) To show: aca) If \( v_1 + K, v_2 + K \in V/K \) then
\[ \hat{T}(v_1 + K) + \hat{T}(v_2 + K) = \hat{T}((v_1 + K) + (v_2 + K)). \]

acb) If \( c \in F \) and \( v + K \in V/K \) then \( \hat{T}(c(v + K)) = c\hat{T}(v + K) \).

aca) Let \( v_1 + K, v_2 + K \in V/K \).
Since \( T \) is a homomorphism,
\[ \hat{T}(v_1 + K) + \hat{T}(v_2 + K) = T(v_1) + T(v_2) \]
\[ = T(v_1 + v_2) \]
\[ = \hat{T}((v_1 + v_2) + K) \]
\[ = \hat{T}((v_1 + K) + (v_2 + K)). \]

acb) Let \( c \in F \) and \( v + K \in V/K \).
Since \( T \) is a homomorphism,
\[ \hat{T}(c(v + K)) = \hat{T}(cv + K) \]
\[ = T(cv) \]
\[ = cT(v) \]
\[ = c\hat{T}(v + K). \]

So \( \hat{T} \) is a linear transformation.

So \( \hat{T} \) is a well defined injective linear transformation.

b) To show: ba) \( T' \) is well defined.

bb) \( T' \) is surjective.

bc) \( T' \) is a linear transformation.

ba) and bb) are proved in Ex. 2.2.3 b), Part I.

bc) To show: bca) If \( v_1, v_2 \in V \) then \( T'(v_1 + v_2) = T'(v_1) + T'(v_2) \).

bcb) If \( c \in F \) and \( v \in V \) then \( T'(cv) = cT'(v) \).

bca) Let \( v_1, v_2 \in V \).
Then, since \( T \) is a linear transformation,
\[ T'(v_1 + v_2) = T(v_1) + T(v_2) = T(v_1) + T'(v_2). \]

bcb) Let \( v_1, v_2 \in V \).
Then, since \( T \) is a linear transformation,
\[ T'(cv) = T(cv) = cT(v) = cT'(v). \]

So \( T' \) is a linear transformation.

So \( T' \) is a well defined surjective linear transformation.

c) Let \( K = \ker T \).
By a), the function

\[ \hat{T}: \ V/K \to W \\
    v + K \to T(v) \]

is a well defined injective linear transformation.

By b), the function

\[ \hat{T}': \ V/K \to \text{im} \hat{T} \\
    v + K \to \hat{T}(v + K) = T(v) \]

is a well defined surjective linear transformation.

To show: ca) \( \text{im} \hat{T} = \text{im} T \).

\text{cb)} \( \hat{T}' \) is injective.

ca) To show: caa) \( \text{im} \hat{T} \subseteq \text{im} T \).

\text{cab)} \( \text{im} T \subseteq \text{im} \hat{T} \).

caa) Let \( w \in \text{im} \hat{T} \).

Then there is some \( v + K \in V/K \) such that \( \hat{T}(v + K) = w \).

Let \( v' \in v + K \).

Then \( v' = v + k \) for some \( k \in K \).

Then, since \( T \) is a linear transformation and \( T(k) = 0 \),

\[ T(v') = T(v + k) \]
\[ = T(v) + T(k) \]
\[ = T(v) \]
\[ = \hat{T}(v + k) \]
\[ = w. \]

So \( w \in \text{im} T \).
So \( \text{im} \hat{T} \subseteq \text{im} T \).

cab) Let \( w \in \text{im} T \).

Then there is some \( v \in V \) such that \( T(v) = w \).

So \( \hat{T}(v + K) = T(v) = w \).

So \( w \in \text{im} \hat{T} \).
So \( \text{im} T \subseteq \text{im} \hat{T} \).

So \( \text{im} T = \text{im} \hat{T} \).

\text{cb)} To show: If \( \hat{T}'(v_1 + K) = \hat{T}'(v_2 + K) \) then \( v_1 + K = v_2 + K \).

Assume \( \hat{T}'(v_1 + K) = \hat{T}'(v_2 + K) \).

Then \( \hat{T}(v_1 + K) = \hat{T}(v_2 + K) \).

Then, since \( T \) is injective, \( v_1 + K = v_2 + K \).

So \( \hat{T}' \) is injective.

Thus we have

\[ \hat{T}': \ V/K \to \text{im} \hat{T} \\
    v + K \to T(v) \]

is a well defined bijective linear transformation. \( \square \)