1 Numbers

The positive integers is the set
\[ \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\} \]
with the operation \( \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \)
\( (i, j) \mapsto i + j \)

The nonnegative integers is the set
\[ \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\} \]
with the operation \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \)
\( (i, j) \mapsto i + j \)

The advantage of the nonnegative integers \( \mathbb{Z}_{\geq 0} \) over the positive integers \( \mathbb{Z}_{>0} \) is that \( \mathbb{Z}_{\geq 0} \) contains an identity element for the operation and \( \mathbb{Z}_{>0} \) does not.

The integers is the set
\[ \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \]
with the operations
\[ \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \]
\( (i, j) \mapsto i + j \)

The advantage of the integers \( \mathbb{Z} \) over the nonnegative integers \( \mathbb{Z}_{\geq 0} \) is that every element of \( \mathbb{Z} \) has an inverse; this is not true in \( \mathbb{Z}_{\geq 0} \). There is another operation on \( \mathbb{Z} \),
\[ \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \]
\( (i, j) \mapsto i + (-j) \)
but this operation is not very well behaved: it is not associative and not commutative (though it does have an identity). There is another operation on \( \mathbb{Z} \)
\[ \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \]
\( (i, j) \mapsto ij \)
and this operation is associative, commutative and has an identity but does not have inverses.

The rationals is the set
\[ \mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} \]
where \( \frac{a}{b} = \frac{c}{d} \) if \( ad = bc \),
with operations defined by
\[ \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \]

The advantage of the rationals \( \mathbb{Q} \) over the integers \( \mathbb{Z} \) is that the multiplication has inverses; well, . . . almost has inverses—the element 0 does not have an inverse.

By long division, every rational number \( \frac{a}{b} \) can be represented as a decimal expansion
\[ d_r d_{r-1} \cdots d_1 d_0. d_{-1} d_{-2} d_{-3} \cdots \]
where the idea is that
\[ d_r d_{r-1} \cdots d_1 d_0. d_{-1} d_{-2} d_{-3} \cdots = \sum_{\ell \in \mathbb{Z}, \ell \leq r} d_\ell 10^\ell. \]

If \( a = a_r \cdots a_1 a_0. a_{-1} a_{-2} \cdots \) is a decimal expansion let \( a_{\leq n} \) be the element of \( \mathbb{Q} \) given by
\[ a_{\leq n} = a_r \cdots a_1 a_0. a_{-1} a_{-2} \cdots a_{-(n-1)} a_{-n}. \]

The real numbers is the set \( \mathbb{R} \) of decimal expansions
\[ \mathbb{R} = \{ d_r \cdots d_1 d_0. d_{-1} d_{-2} \cdots \mid d_i \in \{0, 1, 2, \ldots, 9\} \} \]
with
\[ a = b \quad \text{for all } n \in \mathbb{Z}_{>0} \ (a_{\leq n} - b_{\leq n})_{\leq n-1} = 0 \text{ in } \mathbb{Q}, \]
and operations determined by
\[ a + b = c \quad \text{if, for all } n \in \mathbb{Z}_{>0}, \ (a_{\leq n} + b_{\leq n})_{\leq n-1} = c_{\leq n-1} \text{ in } \mathbb{Q}, \]
and
\[ ab = c \quad \text{if, for all } n \in \mathbb{Z}_{>0}, \ (a_{\leq n} b_{\leq n})_{\leq n-1} = c_{\leq n-1} \text{ in } \mathbb{Q}. \]

An irrational number is a real number that is not a rational number.

**Theorem 1.1.** \( \mathbb{Q} = \{ \text{decimal expansions that repeat} \} \)

**Theorem 1.2.** Irrational numbers exist.

The complex numbers is the set
\[ \mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \} \]
with operations given by
\[ (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i \quad \text{and} \]
\[ (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i. \]

**Theorem 1.3.** (The fundamental theorem of algebra) If \( p_0, p_1, \ldots, p_d \in \mathbb{C} \) with \( p_d \neq 0 \) then there are \( \lambda_1, \ldots, \lambda_d \in \mathbb{C} \) such that
\[ p_0 + p_1 x + p_2 x^2 + \cdots + p_dx^d = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_d). \]

The algebraic numbers is the set
\[ \overline{\mathbb{Q}} = \{ z \in \mathbb{C} \mid \text{there exists } p(x) \in \mathbb{Q}[x], p(x) \neq 0, \text{ with } p(z) = 0 \}. \]

A transcendental number is a complex number that is not an algebraic number.
References


