1 Interiors and closures

Let $X$ be a topological space and let $x \in X$. A neighborhood of $x$ is a subset $N$ of $X$ such that there exists an open subset $U$ of $X$ with $x \in U$ and $U \subseteq N$.

Let $X$ be a topological space and let $E \subseteq X$. A neighborhood of $E$ is a subset $N$ of $X$ such that there exists an open subset $U$ of $X$ with $E \subseteq U \subseteq N$.

Let $X$ be a topological space and let $E \subseteq X$. The interior of $E$ is the subset $E^\circ$ of $E$ such that

(a) $E^\circ$ is open in $X$,
(b) If $U$ is an open subset of $E$ then $U \subseteq E^\circ$.

Let $X$ be a topological space and let $E \subseteq X$. The closure $\overline{E}$ of $E$ is the subset $\overline{E}$ of $X$ such that

(a) $\overline{E}$ is closed,
(b) If $V$ is a closed subset of $X$ and $V \supseteq E$ then $V \supseteq \overline{E}$.

Let $X$ be a topological space. Let $E \subseteq X$. An interior point of $E$ is a point $x \in X$ such that there exists a neighborhood $N_x$ of $x$ with $N_x \subseteq E$.

Let $X$ be a topological space. Let $E \subseteq X$. A close point to $E$ is a point $x \in X$ such that if $N_x$ is a neighborhood of $x$ then $N_x$ contains a point of $E$.

**Theorem 1.1.** Let $X$ be a topological space. Let $E \subseteq X$.

(a) The interior of $E$ is the set of interior points of $E$.
(b) The closure of $E$ is the set of close points of $E$.

2 Hausdorff spaces

A Hausdorff space is a topological space $X$ such that if $x, y \in Y$ and $x \neq y$ then there exist a neighborhood $N_x$ of $x$ and a neighborhood $N_y$ of $y$ such that $N_x \cup N_y = \emptyset$.

**Theorem 2.1.** Let $X$ be a topological space. Show that the following are equivalent:
(a) Any two distinct points of $X$ have disjoint neighborhoods.

(b) The intersection of the closed neighborhoods of any point of $X$ consist of that point alone.

(c) The diagonal of the product space $X \times X$ is a closed set.

(d) For every set $I$, the diagonal of the product space $Y = X^I$ is closed in $Y$.

(e) No filter on $X$ has more than one limit point.

(f) If a filter $\mathcal{F}$ on $X$ converges to $x$ then $x$ is the only cluster point of $x$.

3 Limit points and cluster points

Theorem 3.1. Let $X$ be a topological space and let $(x_1, x_2, \ldots)$ be a sequence in $X$. Then

(a) $y$ is a limit point of $(x_1, x_2, \ldots)$ if and only if, if $N_y$ is a neighborhood of $y$ then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $x_n \in N_y$ for all $n \in \mathbb{Z}_{\geq 0}$, $n \geq n_0$.

(b) $y$ is a cluster point of $(x_1, x_2, \ldots)$ if and only if, if $N_y$ is a neighborhood of $y$ and $n_0 \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ with $n \geq n_0$ such that $x_n \in N_y$.

4 Compact sets

Let $X$ be a set. A filter $\mathcal{F}$ on $X$ is convergent if it has a limit point.

Theorem 4.1. Let $X$ be a topological space. The following are equivalent.

(a) Every filter on $X$ has at least one cluster point.

(b) Every ultrafilter on $X$ is convergent.

(c) Every family of closed subsets of $X$ whose intersection is empty contains a finite subfamily whose intersection is empty.

(d) Every open cover of $X$ contains a finite subcover.