1 Fields of fractions

Let $A$ be a commutative ring. A **zero divisor** is an element $a \in A$ such that there exists $b \in A$ such that $b \neq 0$ and $ab = 0$.

A **integral domain** is a commutative ring $A$ with no zero divisors except 0.

Let $A$ be an integral domain. A **field of fractions** of $A$ is the set

$$F = \left\{ \frac{a}{b} \mid a, b \in A, b \neq 0 \right\},$$

with

$$\frac{a}{b} = \frac{c}{d} \quad \text{if} \quad ad = bc,$$

and operations given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{cd}.$$

**Theorem 1.1.** Let $A$ be an integral domain. Let $F$ be the field of fractions of $A$.

(a) The operations on $F$ are well defined and $F$ is a field.

(b) The map

$$\iota: A \longrightarrow F$$

$$a \longmapsto \frac{a}{1}$$

is an injective ring homomorphism.

(c) If $K$ is a field with an injective ring homomorphism $\zeta: A \rightarrow K$ then there is a unique ring homomorphism $\varphi: F \rightarrow K$ such that $\zeta = \varphi \circ \iota$. 