MATH 221  Lecture 1  September 6, 2000

Calculus is the study of
1) Derivatives  3) Applications of Derivatives
2) Integrals    4) Applications of Integrals

A derivative is a creature you put a function into, it chews on it and spits out a different function

$$f \rightarrow \frac{df}{dx}$$

The integral is the derivative backwards:

$$\int \frac{df}{dx} \rightarrow f$$

A function is one down on the food chain.

input number $x$ $\rightarrow f$ $\rightarrow$ output number $f(x)$

Functions take a number as input, chew on it a bit and spit out a number.

The inverse function to $f$ is $f$ backwards

$$x \rightarrow f^{-1}(x)$$

Example

$$x \rightarrow f(x) = x^2$$

The inverse function is

$$x \rightarrow f^{-1}(x)$$

The inverse function is not always a function because there might be some uncertainty about what the inverse function will spit out

$$9 \rightarrow \sqrt[3]{9} \rightarrow 3$$ or $$9 \rightarrow \sqrt[3]{9} \rightarrow -3$$
Numbers are at the very bottom of the food chain.

Numbers

At some point humankind wanted to count things and discovered the positive integers

\[ 1, 2, 3, 4, 5, 6, \ldots \]

Great for counting something.

But what if you don’t have anything i.e.

nothing, null, zip

and so we discovered the nonnegative integers

\[ 0, 1, 2, 3, 4, 5, \ldots \]

Great for adding \( 1 + 3 = 4, 5 + 0 = 5, 9 + 16 = 25 \).

But not so great for subtracting \( 1 - 3 = ??? \)

and so we discovered the integers

\[ \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \]

Great for adding, subtracting and multiplying.

\[ -2.4 = -8, \quad \pi \cdot 6 = 42, \quad (-7)(-6) = 42 \]

But not so great if you only want part of the sausage...

... and so we discovered the rational numbers

\[ \frac{a}{b}, \quad a \text{ an integer}, \ b \text{ an integer}, \ b \neq 0. \]

Great for addition, subtraction, multiplication and division.

But not so great for finding \( \sqrt{2} \).

... and so we discovered the real numbers

all finite and infinite decimal expansions.

Examples:

\[ \sqrt{2} = 1.414\ldots \]
\[ e = 2.71828\ldots \]
\[ \pi = 3.1415926\ldots \]
\[ \frac{1}{6} = 0.1666\ldots \]
\[ \frac{1}{3} = 0.125 = 0.125000\ldots \]

Great for addition, subtraction, multiplication, and division.

But not so great for finding \( \sqrt{-9} \).
and so we discovered the complex numbers \(a + bi\), \(a\) a real number, \(b\) a real number, 
\[i = \sqrt{-1}.\]

Examples:
\[
\begin{align*}
3+4i &\quad 0+10i = 10i \\
7+9i &\quad -7+9i = -7+9i \\
3.2+6.7i &\quad \frac{1}{2} + \frac{3}{2}i = \frac{1}{2} + \frac{3}{2}i \\
5+0i = 5 &\quad \sqrt{25} + 0i
\end{align*}
\]

and
\[\sqrt{13} = 3^2 + 2^2 = 9 + 4 = 13. \text{ So } \sqrt{13} = 3i.\]

GREAT.

Addition: 
\[(3+4i) + (7+9i) = 10 + 13i\]

Subtraction: 
\[(3+4i) - (7+9i) = 3 - 7 + 4i - 9i = -4 - 5i\]

Multiplication: 
\[(3+4i)(7+9i) = 3(7+9i) + 4i(7+9i) \]
\[= 21 + 27i + 28i + 36i^2 \]
\[= 21 + 27i + 28i - 36 \]
\[= -15 + 55i.\]

Division: 
\[
\frac{3+4i}{7+9i} = \frac{(3+4i)(7-9i)}{(7+9i)(7-9i)}
\]

\[
= \frac{21 - 36i + 28i - 36i^2}{49 - 81i^2} = \frac{57 + i}{130}
\]

= \[\frac{57}{130} + \frac{1}{130}i\]

Square roots: \[\sqrt{3+4i} = i(1+2i)\]

since \((1+2i)^2 = 1 + 2i + 2i + 4i^2 = 1 + 4i - 4 = -3 + 4i;\]

and \(-(-1+2i)^2 = (1+2i)^2 = -3 + 4i\)

Another way is: \[\sqrt{3+4i} = a + bi\]

\[a - 3 + 4i = (a+bi)^2 = a^2 - b^2 + 2abi + bi^2\]

\[= a^2 - b^2 + 2abi\]

So \(a^2 - b^2 = 3\) and \(2ab = 4.\)

Solve for \(a\) and \(b.\)

\[
b = \frac{2}{a} = \frac{2}{a}. \text{ So } a^2 - \left(\frac{2}{a}\right)^2 = -3 \]

\[\text{So } a^2 - \frac{4}{a^2} = -3 \]

\[\text{So } a^2 - 3a^2 = -3 \]

\[\text{So } a^2 - 3a^2 = -3a^2 \]

\[\text{So } a^2 + 3a^2 - 4 = 0 \]

\[\text{So } (a^2 + 4)(a^2 - 1) = 0 \]

\[\text{So } a^2 - 4 \text{ or } a^2 = 1.\]
So $a = \pm 1$. So $b = \pm \frac{2}{1} = 2$ or $-2$.

So $a + bi = 1 + 2i$ or $a + bi = -1 - 2i$.

So $\sqrt{3 + 4i} = \pm (1 + 2i)$.

**Graphing**

3+4i

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Factorizing $x^2 + 5 = (x + \sqrt{5}i)(x - \sqrt{5}i)$

$(x^2 + x + 1) = (x - (-1 + \sqrt{3}))(x - (-1 - \sqrt{3}))$

$= (x - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i))(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i))$

**The Fundamental theorem of algebra is one reason why the complex number system is the right number system to use. It says that any polynomial can be factored completely as**

$(x - u_1)(x - u_2)(x - u_3)\cdots(x - u_n)$ where $u_1, u_2, \ldots, u_n$ are some complex numbers.