Homework 14: Due December 13, 2007

To grade: 13, 14, 15, 16.

1. Define the following terms.
   - vector space
   - subspace
   - span(S)
   - linear combination
   - linearly independent
   - basis
   - linear transformation
   - kernel (of a linear transformation)
   - image (of a linear transformation)
   - eigenvector with eigenvalue \( \lambda \)

2. Define \( M_n(F) \) and appropriate operations and prove that it is a ring.

3. Define \( M_{p\times m}(F) \) and appropriate operations/actions and prove that it is a vector space.

4. Define \( F^n \) and prove that it is a vector space.

5. Let \( V \) be a vector space with basis \( \{s_1, \cdots, s_n\} \). Prove that \( V \cong F^n \).

6. Let \( V \) and \( W \) be vector spaces and let \( \text{Hom}(V, W) \) be the set of linear transformations from \( V \) to \( W \). Define appropriate operations/actions and prove that \( \text{Hom}(V, W) \) is a vector space.

7. Let \( V \) be a vector space. Let \( \text{End}(V) \) be the set of linear transformations from \( V \) to \( V \). Define appropriate operations and prove that \( \text{End}(V) \) is a ring.

8. Let \( V \) be a vector space with basis \( \{s_1, \cdots, s_n\} \). Prove that \( \text{End}(V) \cong M_n(F) \) (as rings).

9. Let \( W \) be a vector space with basis \( \{t_1, \cdots, t_m\} \) and let \( V \) be a vector space with basis \( \{s_1, \cdots, s_n\} \). Prove that \( \text{Hom}(V, W) \cong M_{n\times m}(F) \) (as vector spaces).
10. Let \( f : V \to W \) be a linear transformation. Show that \( f \) is injective if and only if \( \ker f = 0 \).

11. Let \( V \) be a finite dimensional vector space and let \( f : V \to V \) be a linear transformation. Show that \( f \) is invertible if and only if \( \ker f = 0 \).

12. Let \( V \) be a finite dimensional vector space and let \( f : V \to V \) be a linear transformation. Let \( \lambda \in \mathbb{F} \). Define the \( \lambda \)-eigenspace \( V_\lambda \) of \( f \). Find a linear transformation \( h : V \to V \) such that \( \ker h = V_\lambda \).

13. Define the determinant.
   (a) Prove that the determinant is a monoid homomorphism.
   (b) Prove that the determinant is a group homomorphism.
   (c) Prove that the determinant is a ring homomorphism.

14. Write a formula for \( \det(A) \) which corresponds to Laplace expansion down the first column. Prove this formula. Interpret this formula in terms of cosets.

15. Let \( A \) be an \( n \times n \) matrix. Prove that \( A \) is invertible if and only if \( \det(A) \) is invertible.

16. Let \( W \) be a vector space with basis \( \{ t_1, \ldots, t_m \} \) and let \( V \) be a vector space with basis \( \{ s_1, \ldots, s_n \} \). Let \( \varphi_S : V \to \mathbb{F}^n \), \( \varphi_T : W \to \mathbb{F}^m \) and \( \Phi : \text{Hom}(V, W) \to M_{n \times m}(\mathbb{F}) \) be the corresponding isomorphisms (see problem 5 and problem 13). Let \( f : V \to V \) be a linear transformation. Prove that
   (a) \( \varphi_S(\ker f) \) is equal to the null space of \( \Phi(f) \).
   (b) \( \varphi_T(\text{im } f) \) is equal to the column space of \( \Phi(f) \).
   If you give proof machine definitions of null space and column space before beginning these proofs this problem is not difficult but if you don't it is impossible (and probably doesn't make any sense).