Homework 3: Due September 26, 2007

1. Define abelian group and give an example of an abelian group and a group which is not abelian.

2. Define the symmetric groups $S_n$.

3. Define cyclic group. What are the cyclic groups? What is the cardinality of the smallest group which is not cyclic?

4. Define the dihedral groups $D_n$.

5. Define the Klein 4 group.

6. Define the product of two groups. What is the smallest nontrivial example?

7. Define the general linear group, the special linear group, the orthogonal group, the special orthogonal group, the unitary group, the special unitary group, and the symplectic group.

8. Define permutation matrix.

9. Explain how the symmetric group $S_n$ acts on a set with $n$ elements, how the cyclic group $S_n$ acts on an $n$-gon, how the dihedral group acts on an $n$-gon, and how the general linear group acts on an $n$-dimensional vector space.

10. Define subgroup and give some examples.

11. Define cardinality.

12. Define finite, infinite, countable and uncountable.

13. Show that $\text{Card}(\mathbb{Z}_{>0}) = \text{Card}(\mathbb{Z}_{\geq 0})$.

14. Show that $\text{Card}(\mathbb{Z}_{\leq 0}) = \text{Card}(\mathbb{Z})$. 
15. Show that Card(\mathbb{Z}) = Card(\mathbb{Q}).

16. Show that Card(\mathbb{Q}) \neq Card(\mathbb{R}).

17. Show that Card(\mathbb{R}) = Card(\mathbb{C}).

18. Define homomorphism and isomorphism for groups and give some examples.

19. Define homomorphism and isomorphism for rings.

20. Define homomorphism and isomorphism for fields.

21. Let \varphi be a group homomorphism. Define the kernel and the image of \varphi and show that they are subgroups.

22. Let G and H be groups and let \varphi : G \rightarrow H be a function such that if \(g_1, g_2 \in G\) then \(\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)\). Show that \(\varphi(1) = 1\).

23. Let G and H be groups and let \varphi : G \rightarrow H be a function such that if \(g_1, g_2 \in G\) then \(\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)\). Show that if \(g \in G\) then \(\varphi(g^{-1}) = \varphi(g)^{-1}\).

24. Let \varphi be a field homomorphism. Show that \varphi is injective.