Homework 6: Due October 17, 2007

To grade: 6, 10, 17.

1. Define group, subgroup, coset, $G/H$ and normal subgroup.

2. Make a list of the groups with $\leq 10$ elements their subgroups and the corresponding $G/H$.

3. Show that $5\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ and explicitly determine the cosets of $5\mathbb{Z}$ in $\mathbb{Z}$.

4. Let $G$ be a group, $H$ a subgroup and let $g \in G$ and $h \in H$. Show that $gH = ghH$.

5. Let $G$ be a group, $H$ a subgroup and let $x, g \in G$. Show that $x \in gH$ if and only if $gH = xH$.

6. Let $G$ be a group and $H$ a subgroup. Show that $G/H$ is a partition of $G$.

7. Let $G$ be a group, $H$ a subgroup and let $g_1, g_2 \in G$. Show that $\text{Card}(g_1H) = \text{Card}(g_2H)$.

8. Let $H$ be a subgroup of a group $G$. Show that $\text{Card}(G) = \text{Card}(G/H)\text{Card}(H)$.

9. Define integral domain and field of fractions and give examples.

10. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that the operation $+ : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ is well defined.

11. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that the operation $\cdot : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ is well defined.

12. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a group.

13. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Define abelian group and show that $\mathbb{F}$ is an abelian group.
14. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a ring.

15. Let $R$ be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a field.

16. Let $H$ be a subgroup of a group $G$. Show that if the operation on $G/H$ given by
   \[(g_1H)(g_2H) = g_1g_2H\]
   is well defined then $H$ is a normal subgroup of $G$

17. Let $H$ be a subgroup of a group $G$. Show that if $H$ is a normal subgroup of $G$ then the operation on $G/H$ given by
   \[(g_1H)(g_2H) = g_1g_2H\]
   is well defined.

18. Let $H$ be a normal subgroup of a group $G$. Show that $G/H$ with operation given by
   \[(g_1H)(g_2H) = g_1g_2H\]
   is a group.

19. Show that every subgroup of an abelian group is normal.

20. Let $f : G \rightarrow H$ be a group homomorphism. Show that $\ker f$ is a subgroup of $G$.

21. Let $f : G \rightarrow H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of $G$.

22. Let $f : G \rightarrow H$ be a group homomorphism. Show that $\text{im } f$ is a subgroup of $H$.

23. Let $f : G \rightarrow H$ be a group homomorphism. Show that $\text{im } f$ is a normal subgroup of $H$. 
