Homework 8: Due November 1, 2007

To grade: 4, 6, 11.

1. Let \( \mathbb{D} \) be a division ring. Show that the ideals of \( \mathbb{D} \) are \{0\} and \( \mathbb{D} \).

2. Let \( \mathbb{F} \) be a field. Show that the ideals of \( M_n(\mathbb{F}) \) are \{0\} and \( M_n(\mathbb{F}) \).

3. Show that each ideal of \( \mathbb{Z} \) is generated by one element.

4. Show that each ideal of \( \mathbb{R}[x] \) is generated by one element.

5. Give an example of a ring \( R \) and an ideal \( I \) such that \( I \) is not generated by one element (in any possible way). Be sure to prove that \( I \) is not generated by one element.

6. Show that \( (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z} \) as groups.

7. Show that the product of groups \( (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \) is not isomorphic to the group \( \mathbb{Z}/4\mathbb{Z} \).

8. Show that \( \mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C} \).

9. Let \( H \) be a subgroup of a group \( G \). The canonical injection is the map \( \iota : H \rightarrow G \) given by

\[
\iota : H \rightarrow G \\
h \mapsto h
\]

Show that \( \iota : H \rightarrow G \) is a well defined injective group homomorphism.

10. Let \( N \) be a normal subgroup of a group \( G \). The canonical surjection or canonical projection is the map \( \pi : G \rightarrow G/N \) given by
\[ \pi : \ G \longrightarrow \ G / N \]
\[ g \mapsto gN \]
Show that \( \pi : G \rightarrow G / N \) is a well defined surjective group homomorphism and that \( \text{im} \ \pi = G / N \) and \( \ker \pi = N \).

11. Using the notations of problem 10, let \( M \) be a subgroup of \( G \). Show that
   1. \( M / N = \{mN \mid m \in M\} \) is a subgroup of \( G / N \).
   2. \( M / N \) is a normal subgroup of \( G / N \) if \( M \) is a normal subgroup of \( G \).
   3. \( M / N = \pi(M) \) and if \( M \) contains \( N \) Then \( \pi^{-1}(\pi(M)) = M \).
   4. Conclude that there is a one-to-one correspondence between subgroups of \( G \) containing \( N \) and subgroups of \( G / N \).
   5. Show that this correspondence takes normal subgroups to normal subgroups.

12. Let \( I \) be an ideal of a ring \( R \). The canonical injection is the map \( \iota : I \rightarrow R \) given by
   \[ \iota : \ I \longrightarrow \ R \]
   \[ i \mapsto i \]
Show that \( \iota : I \rightarrow R \) is a well defined injective ring homomorphism.

13. Let \( I \) be an ideal of a ring \( R \). The canonical surjection or canonical projection is the map \( \pi : R \rightarrow R / I \) given by
   \[ \pi : \ R \longrightarrow \ R / I \]
   \[ r \mapsto r + I \]
Show that \( \pi : R \rightarrow R / I \) is a well defined surjective homomorphism and that \( \text{im} \ \pi = R / I \) and \( \ker \pi = I \).

14. Using the notations of problem 13, let \( J \) be an ideal of \( R \). Show that
   1. \( J / I = \{j + I \mid j \in I\} \) is an ideal of \( R / I \).
   2. \( J / I = \pi(J) \) and if \( J \) contains \( I \) then \( \pi^{-1}(\pi(J)) = J \).
   3. Conclude that there is a one-to-one correspondence between ideals of \( R \) containing \( I \) and ideals of \( R / I \).