When push comes to shove

Axel Almet

Supervisors: Barry Hughes & Kerry Landman

Department of Mathematics and Statistics, University of Melbourne

Introduction

Exclusion processes are interacting random walk models that take place on a lattice, where agents can move and each site can be occupied by at most one agent [1]. They can be used in a widespread number of applications, such as cell motility, traffic flow, or crowd movement. We can extract information about the individual or average collective behaviour from these models.

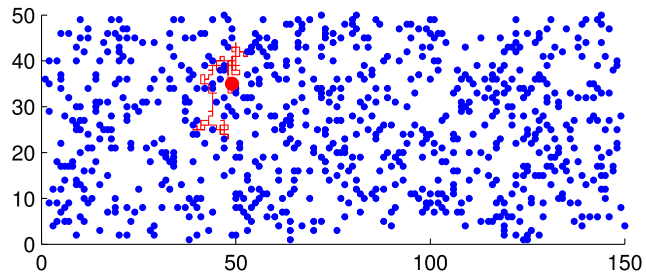

Figure 1: *Tagged agent in a crowd.*

One of the most used exclusion processes is simple exclusion. Under a *symmetric* simple exclusion process in one dimension, each agent has a probability P of attempting a move, and can move left or right with equal chance. If an agent attempts to move to a site that is already occupied by an agent, it will abort its move.

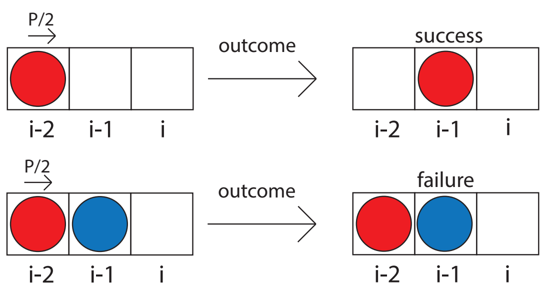

Figure 2: *Interaction rules under simple exclusion.*

The average behaviour of this process can be accurately described [2] by the linear diffusion equation

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial x^2},$$

where D_0 is the diffusivity constant.

An alternative: shoving

We wish to develop a motility mechanism in which attempted moves are always successful, whilst maintaining exclusion. We achieved this through *shoving*. Under the shoving process, agents follow the exact same movement rules as for simple exclusion. However, this time, when agent attempts to move to an occupied site, the agent occupying the target site will be shoved in the same direction as the motile agent, as shown in figure 3.

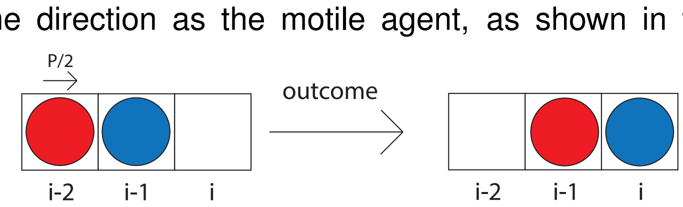

Figure 3: *An agent moving from site $i-2$ under the shoving process.*

Under this new process, interactions are now nonlocal. We can have agents an arbitrary number of sites apart affecting each other through a domino-type effect. This is a key factor in deriving the PDE for this process.

Modelling

We derive a PDE describing the average density for this process. Let $C_i(t)$ denote the average occupancy. We say that an agent is at site i at time $t = k\tau$ if $x = i\Delta$, where $i \in \mathbb{Z}$, Δ is the lattice spacing and τ is one time increment. Assume that the average occupancies of sites are independent.

The change in average occupancy between times $t+\tau$ and t is given by

$$\begin{aligned} C_i(t+\tau) - C_i(t) &= \frac{P}{2}[1 - C_i(t)] \sum_{k=1}^{\infty} \prod_{l=1}^{k} C_{i-l}(t) \\ &+ \frac{P}{2}[1 - C_i(t)] \sum_{k=1}^{\infty} \prod_{l=1}^{k} C_{i+l}(t) \\ &- \frac{P}{2} C_i(t) - \frac{P}{2} C_i(t). \end{aligned}$$

Keeping the first n terms of each sum, taking the 2^{nd}-order Taylor expansions, neglecting terms higher than $O(\Delta^2)$, dividing both sides by τ then taking the limits $\Delta \to 0$, $\tau \to 0$, $n \to \infty$, we obtain

$$\begin{aligned} \frac{\partial C}{\partial t} &= D_0 \frac{\partial}{\partial x}\left[\frac{1+C}{(1-C)^3}\frac{\partial C}{\partial x}\right], \\ D_0 &= \lim_{\Delta,\tau \to 0}\left(\frac{P\Delta^2}{2\tau}\right). \end{aligned}$$

Results and more

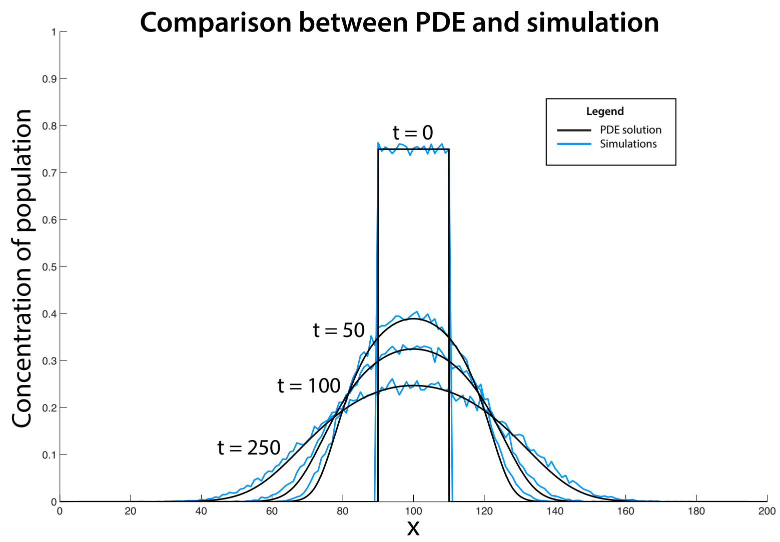

Figure 4: *Initial density = 0.75*

We averaged over 2500 realisations to obtain density profiles and used the MATLAB routine pdepe.m to solve this PDE. As shown in figure 4, the PDE matches simulation results very well, confirming that it correctly describes the process.

We also derived PDEs for a multi-species equivalent of this process and for a 'mixed exclusion' process, consisting of both simple exclusion and shoving.

Experience

The vacation scholarship was a very rewarding experience that granted me excellent insight into the world of mathematics research. It seems that most research starts from mere curiosity and builds from there. I was also exposed to the unpredictable pace of research: days spent trying to overcome a problem, be it analysis or programming related, until that lightbulb moment suddenly hits you.

I have to extend my utmost gratitude and thanks to my supervisors, Barry Hughes and Kerry Landman, for allowing me to learn so much about research in mathematical biology and for their guidance. Finally, my thanks to the University of Melbourne Department of Maths and Stats for giving me this opportunity and for allowing me to drink liberal amounts of the faculty tea and coffee with impunity.

References

[1] T.M. Liggett, Stochastic Interacting Systems: Contact, Voter and Exclusion Processes, Springer-Verlag, Berlin, 1999.

[2] M.J. Simpson, K.A. Landman, B.D. Hughes, AJEE 15 (2009) 59-67.