DYER-LASHOF OPERATIONS IN THE STRING TOPOLOGY OF SPHERES AND PROJECTIVE SPACES

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Abstract. We compute the 2-primary Dyer-Lashof operations in the string topology of several families of manifolds, specifically spheres and a variety of projective spaces. These operations, while well known in the context of iterated loop spaces, give a collection of homotopy invariants of manifolds new to string topology. The computations presented here begin an exploration of these invariants.

1. Introduction

In [CS01], Chas and Sullivan introduced a new collection of invariants of manifolds under the moniker "String topology". Specifically, they geometrically constructed a product (the *loop product*) on the *loop homology*, $\mathbb{H}_*(M) := H_{*-n}(LM)$ of the free loop space of an n-manifold M. Using the homotopy commutativity of this product on the chain level, they also defined a Lie bracket – the *loop bracket* $\{\cdot,\cdot\}$ – on $\mathbb{H}_*(M)$. Together, these operations give the loop homology the structure of a Gerstenhaber algebra.

Using a purely algebraic construction coming from Hochschild cohomology, one may define a similar collection of invariants in $\mathbb{H}_*(M)$. Some (products and brackets) are expected to be the same as Chas and Sullivan's, and some have yet to be considered in string topology (Dyer-Lashof operations). Our goal in this paper is to compute these algebraic invariants for certain manifolds.

Let us be more specific about the origin of these algebraic invariants. In [CJ02], Cohen and Jones realized Chas and Sullivan's structure homotopically in the multiplicative properties of a certain ring spectrum LM^{-TM}, via a Thom isomorphism $\mathbb{H}_*(M) \cong H_*(LM^{-TM})$. Additionally, when M is simply connected, they identified the latter homology as the Hochschild cohomology of the singular cochains of M:

$$H_*(LM^{-TM}) \cong HH^*(C^*(M), C^*(M))$$

In fact, this holds for certain non-simply connected cases, such as $\mathbb{R}P^n$, as we will demonstrate.

Deligne conjectured that the Hochschild cochain complex of an associative algebra R, $CH^*(R,R)$, admits the structure of an algebra over the little disks operad, C_2. This conjecture was proven by a number of authors [MS02, Tam98a, Tam98b, Vor00, KS00]. This structure is also constructed homotopically for LM^{-TM} in [CJ02]. F. Cohen's work in [CLM76] shows that the homology of a C_2-algebra is a Gerstenhaber algebra.

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So, in sum, the three chain complexes,

$$C_*(LM),\ C_*(LM^{-TM}),\ CH^*(C^*(M), C^*(M)),$$

(which are all quasi-isomorphic, via [CJ02]) each admit certain similar algebraic structures. The first gives rise to a Gerstenhaber algebra through explicit geometric constructions, and the latter two are C_2-algebras, the second from a homotopical construction, and the third through an algebraic construction. One wonders whether all three Gerstenhaber structures agree or, to be more ambitious, whether the the latter two are homotopy equivalent as C_2-algebras. To our knowledge, none of these questions have been answered, though it has been shown in [Coh04] that the loop product of [CJ02] and Hochschild product agree.

For the purposes of this paper, we will employ the C_2-structure coming from McClure and Smith's proof [MS02] of Deligne's conjecture. That is, our results will be phrased as statements about $HH^*(C^*(M), C^*(M))$, rather than $\mathbb{H}_*(M)$ or $H_*(LM^{-TM})$, quietly hoping that the three Gerstenhaber structures agree.

In [CS01] the loop product and bracket were constructed integrally – that is, on $\mathbb{H}_*(M;\ \mathbb{Z})$. At a prime p, however, the homology $H_*(X;\ \mathbb{F}_p)$ of a C_2-algebra X is endowed with further structure: the Dyer-Lashof and Browder operations, originally introduced in the study of iterated loop spaces. To use the notation of [CLM76], there are unary operations Q_0, Q_1 and a binary operation λ_1. The zeroth Dyer-Lashof operation Q_0 is just the p^{th} power map with respect to the product. The Browder operation λ_1 is a Lie bracket constructed from a homotopy for commutativity of X. If the Hochschild and homotopy theoretic actions of C_2 on $\mathbb{H}_*(M)$ do in fact concide, then λ_1 is just the reduction mod p of the loop bracket, $\{\cdot,\cdot\}$. The first Dyer-Lashof operation Q_1 is, like the Browder operation, a byproduct of homotopy commutativity. The operations Q_0, Q_1, and λ_1 on $HH^*(C^*(M), C^*(M))$ are what is meant by the term "algebraic invariants" above.

The purpose of this paper is to give concrete, nontrivial examples of such structure. Specifically, we compute the 2-primary operations in $HH^*(C^*(M), C^*(M))$ of four families of manifolds M: spheres and real, complex, and quaternionic projective spaces. Though the ring structure of $\mathbb{H}_*(M)$ has been explored for a variety of manifolds (for instance, [CS01, CJY02, Abb03]), there has been little discussion of the Dyer-Lashof operations. While Q_0 is familiar in the guise of the p^{th} power map and λ_1 is conjecturally the loop bracket, Q_1 has not been studied in the context of string topology. Since the quasi-isomorphism type of $C^*(M)$ as a differential graded algebra is a homotopy invariant of M, so too are $HH^*(C^*(M), C^*(M))$ and all invariants such as products, Q_i, and λ_1 derived from that DGA structure.

Without other decoration, H_* will denote homology with coefficients in the field of two elements, $\mathbb{F}_2$. As we will be discussing quaternionic projective spaces, we will reserve the notation $\mathbb{H}$ for the quaternions, and from now on write $H_*(LM^{-TM})$ or $HH^*(C^*(M), C^*(M))$ for the loop homology of M, after [CJ02]. In the (simpler) case of spheres, we detect a nonzero Browder operation:

Theorem 1.1. *If $k > 1$, $HH^*(C^*(S^k), C^*(S^k))$ is isomorphic as an algebra to $\mathbb{F}_2[x, v]/(x^2)$, where the dimensions of x and v are $-k$ and $k-1$, respectively. The Dyer-Lashof structure is determined by three equations: $Q_1(x) = 0$, $Q_1(v) = 0$, and $\lambda_1(x, v) = 1$.*

There are a number of relationships between the Browder and Dyer-Lashof operations described in detail in [CLM76]. One curious consequence of these relationships and Theorem 1.1 is the fact that

$$Q_1(xv) = xv$$

That is, these C_2-algebras give examples of homology classes for which Q_1 acts as the identity. This oddity also occurs for the projective spaces:

Theorem 1.2. *Let K be one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $d = \dim_{\mathbb{R}}(K)$. For n odd (and greater than 1 if $K = \mathbb{R}$), there is a ring isomorphism*

$$HH^*(C^*(KP^n), C^*(KP^n)) = \mathbb{F}_2[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1})$$

and for n even,

$$HH^*(C^*(KP^n), C^*(KP^n)) = \mathbb{F}_2[x, u, t]/(x^{n+1}, u^2, tx^n, ux^n)$$

where the dimensions of $x, u, v,$ and t are $-d$, -1, $d-1$, and $d(n+1)-2$ respectively. On generators, the Dyer-Lashof operations are given by

$$Q_1(x) = 0, \quad Q_1(t) = 0, \quad Q_1(v) = 0, \quad Q_1(u) = u.$$

The Browder operation λ_1 is given on generators by

$$\lambda_1(x, v) = 1, \quad \lambda_1(x, u) = x, \quad \lambda_1(x, t) = 0, \quad \lambda_1(v, t) = 0, \quad \lambda_1(u, t) = t.$$

Note that the Cartan formulas for 2-primary C_2-algebras

$$Q_1(ab) = a^2Q_1(b) + Q_1(a)b^2 + a\lambda_1(a, b)b$$

$$\lambda_1(a, bc) = \lambda_1(a, b)c + b\lambda_1(a, c)$$

give the entire Dyer-Lashof structure of these rings from the information provided.

The statements about ring structures are not all new. Many may be obtained either directly from or with the obvious alterations to the approach given in [CJY02]. The methods of [CJY02] do not serve to determine the Dyer-Lashof operations, however.

Our approach to these computations is as follows. The goal of Section 2 is a formality result; we show that $HH^*(C^*(M), C^*(M))$ may be identified with $HH^*(H^*(M), H^*(M))$ for all the manifolds M of Theorems 1.1 and 1.2. Using [CJ02], one thereby knows that $H_*(LM^{-TM}) \cong HH^*(H^*(M), H^*(M))$ except for $M = \mathbb{R}P^n$, a detail which is resolved in Section 5. We then compute the ring structure on $HH^*(H^*(M), H^*(M))$ using the description of Hochschild cohomology as a bimodule Ext. This is done in Section 3. These computations are likely to be well known and, in any case, are not difficult; we include them for completeness. The similarities shared amongst the cohomology algebras for spheres and projective spaces allow us to perform these computations simultaneously. The generators of these rings occur in Hochschild degrees $0, 1,$ and 2. The Hochschild cochain complex is tractable in these degrees, so we employ it and Gerstenhaber's ideas [Ger63] on its homotopy commutativity to compute the Dyer-Lashof operations in Section 4.

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2. Formality Considerations

We will grade (singular, simplicial) chain complexes of spaces positively and cochain complexes negatively. Let K be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Proposition 2.1. *At $p=2$, S^k and KP^n are formal. That is, their mod 2 singular cochain algebras are quasi-isomorphic to their cohomology algebras.*

Proof. Let $C_*^{simp}(X)$ denote the simplicial chain complex of a simplicial set X, and $C_*^N(X)$ the normalized quotient complex. It is well known (e.g., [May67]) that $C_*^N(X)$ is quasi-isomorphic to the singular chain complex of the realization of X, $C_*(|X|)$. Moreover, we may endow both $C_*^{simp}(X)$ and $C_*(|X|)$ with a diagonal defined using the Alexander-Whitney map:

$$AW(a) = \sum_{i=0}^{n} d_{n-i}a \otimes d_i' a$$

Here $a \in C_n^{simp}(X)$ or $C_n(|X|)$, $d_{n-i}a$ is its last $n-i$ faces, and $d_i'a$ its first i faces.

One can see that AW carries degenerate simplices to degenerate simplices, so AW is well-defined on $C_*^N(X)$ and the quasi-isomorphism $C_*^N(X) \simeq C_*(|X|)$ respects the coalgebra structure. Since S^k and KP^n may be constructed as realizations M of simplicial sets, their singular cochain algebras $C^*(M)$ are quasi-isomorphic to $C_N^*(M)$, the normalized cochain complex.

The cohomology algebras of all of these manifolds are generated by a single class (in dimension $-k$ for S^k and in dimension $-\dim_{\mathbb{R}} K$ for KP^n). Let x be a cocycle in $C_N^*(M)$ representing that class. This gives a map of DGA's $i : \mathbb{F}_2[x] \to C_N^*(M)$ where $\mathbb{F}_2[x]$ has the zero differential. From the normalization, $C_N^p(M) = 0$ for $p < -\dim M$. Therefore i descends to a map of DGA's

$$i' : \mathbb{F}_2[x]/(x^{n+1}) \to C_N^*(M)$$

(where $n=1$ for S^k). Since $H^*(M) \cong \mathbb{F}_2[x]/(x^{n+1})$, i' is obviously a quasi-isomorphism of DGA's. □

This proposition implies that for all of the manifolds listed,

$$HH^*(C^*(M;\, \mathbb{F}_2), C^*(M;\, \mathbb{F}_2)) \cong HH^*(H^*(M;\, \mathbb{F}_2), H^*(M;\, \mathbb{F}_2))$$

In [CJ02] it was shown that for compact, orientable, simply connected manifolds M,

$$(*) \qquad H_*(LM^{-TM};\, \mathbb{Z}) \cong HH^*(C^*(M;\, \mathbb{Z}), C^*(M;\, \mathbb{Z}))$$

So for $M = S^k$ ($k > 1$) or $M = KP^n$ ($K = \mathbb{C}$ or $\mathbb{H}$), $H_*(LM^{-TM};\, \mathbb{F}_2)$ may be computed as

$$H_*(LM^{-TM};\, \mathbb{F}_2) \cong HH^*(H^*(M;\, \mathbb{F}_2), H^*(M;\, \mathbb{F}_2))$$

We would like the same result to hold for $\mathbb{R}P^n$ ($n > 1$). While compact, these manifolds are not simply connected, nor always oriented. The assumption of orientability for M in $(*)$ is used only to ensure that Poincaré duality holds, and therefore may be discarded by taking coefficients in $\mathbb{F}_2$. The connectivity assumption is more substantial, but we will show in Section 5 that nonetheless

$$H_*(L\mathbb{R}P^{n-T\mathbb{R}P^n};\, \mathbb{F}_2) \cong HH^*(H^*(\mathbb{R}P^n;\, \mathbb{F}_2), H^*(\mathbb{R}P^n;\, \mathbb{F}_2))$$

We therefore obtain the following corollary, which is the starting point of our computations.

Corollary 2.2. *Let $M = S^k$ $(k > 1)$, $\mathbb{R}P^n$ $(n > 1)$, $\mathbb{C}P^n$, or $\mathbb{H}P^n$. Then*

$$H_*(LM^{-TM}) \cong HH^*(H^*(M), H^*(M)) \cong HH^*(C^*(M), C^*(M))$$

3. The Ring Structure

Let k be a commutative ring, and form the graded ring $R = k[x]/(x^{n+1})$ where x is an indeterminate of degree $r \neq 0$. Then the Hochschild cohomology of R is the same as the bimodule Ext:

$$HH^*(R, R) = \mathrm{Ext}^*_{R\otimes R^{op}}(R, R)$$

R is commutative, so $R = R^{op}$. Let $P(y, z) = y^n + y^{n-1}z + \ldots + z^n \in R \otimes R = k[y, z]/(y^{n+1}, z^{n+1})$. The following is a graded version of an exercise in Loday's text [Lod92] (p. 121).

Proposition 3.1. *This is an $R \otimes R$-free resolution of R:*

$$0 \longleftarrow R \xleftarrow{\ \epsilon\ } R\otimes R \xleftarrow{\ y-z\ } \Sigma^r R\otimes R \xleftarrow{\ P(y,z)\ } \Sigma^{r(n+1)} R\otimes R \xleftarrow{\ y-z\ } \ldots$$

the cochains are therefore suspensions of $Hom_{R\otimes R}(R\otimes R, R) = R$ and

1. *$HH^0(R, R) = R$.*
2. *For $i \geq 0$, $HH^{2i+1}(R, R) = \ker(n + 1)x^n \subseteq \Sigma^{-r(1+(n+1)i)}R$.*
3. *For $i > 0$, $HH^{2i}(R, R) = \Sigma^{-r(n+1)i}R/((n + 1)x^n)$.*

Now, and for the rest of this paper, fix $k = \mathbb{F}_2$.

Definition 3.2. *Let $x \in HH^0(R, R)$ correspond to $x \in R$. For n odd, write v for the generator of $HH^1(R, R)$ as an R-module; this corresponds to $1 \in \Sigma^{-r}R$. For n even, write u as the generator of $HH^1(R, R)$ as an R-module; this corresponds to $x \in \ker(n + 1)x^n \subseteq \Sigma^{-r}R$. Finally, let $t \in HH^2(R, R)$ be the generator as an R-module; this corresponds to $1 \in \Sigma^{-r(n+1)}R/((n + 1)x^n)$.*

Denote by $|a|$ the *total* (or *topological*) degree of a:

$$|\cdot| = -(\text{Hochschild degree}) + (\text{suspension degree}) + (\text{internal degree})$$

Then $|x| = r$, $|v| = -1 - r$, $|u| = -1$, and $|t| = -2 - r(n + 1)$

Lemma 3.3. *$u^2 = 0$, and $v^2 = \frac{n+1}{2}tx^{n-1}$*

Lemma 3.4. *Multiplication by t is an isomorphism $HH^i(R, R) \to HH^{i+2}(R, R)$ for $i > 0$.*

These lemmas immediately imply the following corollary:

Corollary 3.5. *If n is odd,*

$$HH^*(R, R) = \mathbb{F}_2[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1})$$

and if n is even,

$$HH^*(R, R) = \mathbb{F}_2[x, u, t]/(x^{n+1}, u^2, tx^n, ux^n)$$

We recover the statements in Theorems 1.1 and 1.2 about ring structure for KP^n by taking $r = -\dim_{\mathbb{R}}(K)$ and for S^k by taking $r = -k$, $n = 1$, since these choices define their cohomology algebras.

For the proofs of both of the lemmas, we use the description of the multiplication in Ext as the composition product.

Proof of Lemma 3.3

The cochain representing v^2 is given by the map $v \circ v_1$ in the diagram below, where v_i are projective lifts of $v \in Hom(\Sigma^r R \otimes R, R)$:

$$\begin{array}{ccccccc}
 & & \Sigma^r R \otimes R & \xleftarrow{P(y,z)} & \Sigma^{r(n+1)} R \otimes R & \xleftarrow{y-z} & \ldots \\
 & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \\
R & \xleftarrow{\epsilon} & R \otimes R & \xleftarrow{y-z} & \Sigma^r R \otimes R & \xleftarrow{P(y,z)} & \ldots \\
 & & & & \downarrow{\scriptstyle v} & & \\
 & & & & R & &
\end{array}$$

Here, $\epsilon \circ v_0 = v$.

Since $v(1) = 1$, v_0 may be taken to be the identity. So v_1 satisfies the equation $v_1(1)(y - z) = P(y, z)$. Hence

$$v_1(1) = y^{n-1} + y^{n-3}z^2 + \ldots + y^2 z^{n-3} + z^{n-1}$$

and so $v \circ v_1(1) = \frac{n+1}{2} x^{n-1}$. Therefore $v^2 = \frac{n+1}{2} t x^{n-1}$.

There is a similar diagram for computing u^2; just replace v's by u's. Here, however, $u(1) = x$, so we may take $u_0(1) = y$. Then $u_1(1)(y - z) = yP(y, z)$. So $u_1(1)$ is homogenous of degree n. Hence $u \circ u_1(1)$ is of degree $n + 1$, and is therefore 0. Thus $u^2 = 0$.

□

Proof of Lemma 3.4

Let f be a generator of $HH^i(R, R)$ as an R-module. We need to show that tf is a generator of $HH^{i+2}(R, R)$. Again, we lift f through the resolution (for simplicity, we ignore suspensions):

$$\begin{array}{ccccccccc}
 & & R \otimes R & \longleftarrow & R \otimes R & \longleftarrow & R \otimes R & \longleftarrow & \ldots \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
R & \xleftarrow[\epsilon]{} & R \otimes R & \xleftarrow[y-z]{} & R \otimes R & \xleftarrow[P(y,z)]{} & R \otimes R & \longleftarrow & \ldots \\
 & & & & & & \downarrow{\scriptstyle t} & & \\
 & & & & & & R & &
\end{array}$$

Here the domain of f_j is the $i + j^{\text{th}}$ term of the resolution, and $\epsilon \circ f_0 = f$. We need to compute $t \circ f_2$.

If i is even, then $f(1) = 1$, and the maps in the upper resolution match up with those in the lower resolution. Hence, we may take f_2 to be the identity. So $t \circ f_2(1) = 1$. Thus tf is the generator of $HH^{i+2}(R, R)$

If i is odd, the maps in the upper resolution are the opposite of those in the lower resolution. If n is odd, then $f(1) = 1$, so f_0 may be taken to be the identity.

Hence

$$(y-z)f_1(1) = P(y,z)$$

and

$$P(y,z)f_2(1) = (y-z)f_1(1)$$

so we may take f_2 to be the identity. As above, this implies that tf is a generator.

Similarly, one may show that if n is even (so that $f(1) = x$), we may take $f_2(1) = y$; this implies that $t \circ f_2(1) = x$, so tf is again a generator.

□

4. The Dyer-Lashof Structure

Dyer-Lashof operations, originally defined on the homology of iterated loop spaces by Araki-Kudo [AK56], Browder [Bro60], and Dyer-Lashof [DL62] were reformulated in [CLM76] and [May70] as operations on the homology of algebras over the little $n+1$-disk operad, C_{n+1}. We recall their construction. For such an algebra X, let θ be the operad action map

$$\theta : C_{n+1}(m) \times_{\Sigma_m} X^{\times m} \to X$$

This allows Cohen, Lada, and May to define the Dyer-Lashof and Browder operations in terms of the homology of $C_{n+1}(m)$. The natural inclusion $C_{n+1}(m) \to C_\infty(m)$ is Σ_m-equivariant, and so induces

$$H_*(C_{n+1}(m) \times_{\Sigma_m} X^{\times m}) \to H_*(C_\infty(m) \times_{\Sigma_m} X^{\times m})$$

To define the operations for $p = 2$, we need only examine the case $m = 2$. The righthand side is the homology of $E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^{\times 2}$. If $\{x_j\}$ is a totally ordered basis for $H_*(X)$, define $A \subseteq H_*(X) \otimes H_*(X)$ as having basis $\{x_j \otimes x_j\}$ and $B \subseteq H_*(X) \otimes H_*(X)$ with basis $\{x_i \otimes x_j \mid i < j\}$. Then it is easy to see that

$$H_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^{\times 2}) = (\bigoplus_{i=0}^{\infty} e_i \otimes A) \bigoplus (e_0 \otimes B)$$

where e_i (of dimension i) corresponds to the generator of $H_i(B\mathbb{Z}/2)$. Since $C_{n+1}(2)$ is the $\mathbb{Z}/2$-equivariant n-skeleton of $C_\infty(2)$, there are classes

$$\bigoplus_{i=0}^{n} e_i \otimes A \subseteq H_*(C_{n+1}(2) \times_{\mathbb{Z}/2} X^{\times 2})$$

which map to the classes of the same name in $H_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^{\times 2})$ under the inclusion above.

Definition 4.1. *For a C_{n+1}-algebra X, $x \in H_*(X)$ and $0 \le i \le n$, the i^{th} Dyer-Lashof operation is defined as*

$$Q_i(x) := \theta_*(e_i \otimes x \otimes x)$$

The top operation Q_n is also denoted ξ_n.

To define the Browder operation, let $\tilde{\theta}$ be the action map

$$\tilde{\theta} : C_{n+1}(2) \times X^{\times 2} \to X$$

It can be shown that $C_{n+1}(2) \simeq S^n$, with fundamental class ι.

Definition 4.2. *For $x, y \in H_*(X)$, where X is a C_{n+1}-algebra, the Browder operation is defined as*

$$\lambda_n(x,y) := \tilde{\theta}_*(\iota \otimes x \otimes y)$$

The singular chain complex of C_{n+1} forms an operad in the category of chain complexes. A chain complex X which is an algebra over the chain version of C_{n+1} admits Dyer-Lashof and Browder operations on its homology in the same fashion as above.

For our applications, we employ the Hochschild cochain complex of an associative algebra A whose n^{th} term is

$$CH^n(A,A) = \text{Hom}_k(A^{\otimes n}, A)$$

In [Ger63], Gerstenhaber introduces a degree -1 binary operation on $CH^*(A,A)$ which he calls $\circ$. To avoid confusion with composition, we will follow Chas and Sullivan's example in [CS01] and denote $\circ$ by $*$. We postpone its definition for a moment and describe its features. The operation $*$ gives a chain homotopy for commutativity: if $f \in CH^m(A,A)$ and $g \in CH^n(A,A)$, then

$$f * (dg) + d(f * g) + (df) * g = gf - fg$$

(we drop Gerstenhaber's signs as we are working mod 2). Here d is the coboundary operator on $CH^*(A,A)$. Gerstenhaber gives $CH^*(A,A)$ the structure of a Lie algebra using $*$; the bracket is defined by $[f,g] = f * g - g * f$.

Explicitly, $*$ is given as follows:

$$f*g(a_1\otimes\ldots\otimes a_{m+n-1}) = \sum_{i=1}^{m} f(a_1\otimes\ldots\otimes a_{i-1}\otimes g(a_i\otimes\ldots\otimes a_{i+n-1})\otimes a_{i+n}\otimes\ldots\otimes a_{m+n-1})$$

(in the case $m = 0$, so that f is a 0-cochain, $f * g = 0$). This homotopy will indicate how to compute Q_1 and λ_1:

Lemma 4.3. *Let $a, b \in HH^*(A,A)$ be represented by $\overline{a}, \overline{b} \in CH^*(A,A)$.*

1. $Q_0(a) = a^2$
2. $Q_1(a)$ *is represented by* $\overline{a} * \overline{a}$.
3. $\lambda_1(a,b)$ *is represented by* $[\overline{a}, \overline{b}] = \overline{a} * \overline{b} - \overline{b} * \overline{a}$.

Proof. The first statement is definitional. In their proof of Deligne's conjecture [MS02], McClure and Smith construct a chain operad $\mathcal{H}$ which is generated by an identity, cup product, and brace operations generalizing Gerstenhaber's $*$, subject to certain relations. $\mathcal{H}$ acts naturally on $CH^*(A,A)$, and Deligne's conjecture is proven by showing that $\mathcal{H}$ is quasi-isomorphic to $C_*(C_2)$. The differential for $\mathcal{H}$, ∂, measures the deviation of an operation ν from being a chain operation:

$$\partial(\nu) = d \circ \nu - \nu \circ d$$

Gerstenhaber's $*$ represents an element of $\mathcal{H}_1(2)$ (the dimension 1 part of the second term of the operad). Its property as a homotopy for commutativity implies that $(\partial(*))(f \otimes g) = gf - fg$. That is, if τ is the twist induced by the $\mathbb{Z}/2$ action, and $\smile$ is the cup product,

$$(\dagger) \qquad \partial(*) = (\smile \circ \tau) - \smile$$

Thus $*$, while not itself a cycle, gives a $\mathbb{Z}/2$-equivariant cycle of dimension 1. It therefore represents e_1 as defined above, and so

$$Q_1(a) = \theta_*(e_1 \otimes a \otimes a) = \overline{a} * \overline{a}$$

Notice that by (†), $* - (* \circ \tau)$ is in fact a cycle in $\mathcal{H}_1(2)$; in particular it is the fundamental cycle ι. Thus

$$\lambda_1(a,b) = \tilde{\theta}(\iota \otimes a \otimes b) = \overline{a} * \overline{b} - \overline{b} * \overline{a}$$

□

Using this description, we will compute a number of Dyer-Lashof and Browder operations. First, we need to find chain representatives of the generators of $HH^*(R,R)$.

Lemma 4.4.

(1) *$x \in HH^0(R,R)$ is represented by $x \in CH^0(R,R) = R$.*
(2) *For n odd, $v \in HH^1(R,R)$ is represented by the function $\overline{v} : R \to R$ which is the k-linear map $x^{2m} \mapsto 0$ and $x^{2m+1} \mapsto x^{2m}$.*
(3) *For n even, $u \in HH^1(R,R)$ is represented by the function $\overline{u} : R \to R$ which is the k-linear map $x^{2m} \mapsto 0$ and $x^{2m+1} \mapsto x^{2m+1}$.*

Proof. Item 1 is obvious. We show that $\overline{v}$ is a cocycle:

$$(d\overline{v})(x^i \otimes x^j) = x^i \overline{v}(x^j) + \overline{v}(x^i)x^j + \overline{v}(x^{i+j})$$

If both i and j are even, then each term vanishes. If both are odd, then the last term vanishes, and the first two terms are equal, so the sum vanishes. Lastly, if one (say i) is odd, and the other even, then

$$(d\overline{v})(x^i \otimes x^j) = x^{i-1}x^j + \overline{v}(x^{i+j}).$$

If $i+j \leq n$, then the last term is x^{i+j-1}, and the two terms cancel. If $i+j > n+1$, then both terms are 0. Finally, since $i+j$ is odd and n is odd, the troublesome case $i+j = n+1$ never occurs. A similar argument shows that $\overline{u}$ is a cocycle.

Since $d : CH^0(R,R) \to CH^1(R,R)$ is zero, $\overline{v}$ and $\overline{u}$ represent nonzero elements in cohomology. We may determine what elements they do represent by examining their topological degree. Since $\overline{v} : R \to R$ lowers degree by r and has Hochschild degree 1, its topological degree is $-1-r$. So $\overline{v}$ does in fact represent v. Similarly, $\overline{u}$ represents u – it does not lower degree, so its topological degree is -1.

□

Lemma 4.5. *The following formulas hold:*

$$Q_1(x) = 0, \quad Q_1(v) = 0, \quad Q_1(u) = u.$$
$$\lambda_1(x,v) = 1, \quad \lambda_1(x,u) = x, \quad \lambda_1(u,t) = t.$$

The remaining operations may be determined using filtration arguments:

Lemma 4.6. *The following formulas hold:*

$$Q_1(t) = 0, \quad \lambda_1(x,t) = 0, \quad \lambda_1(v,t) = 0.$$

Theorems 1.1 and 1.2 now follow from these lemmas and Corollary 3.5.

Proof of Lemma 4.5

That $Q_1(x)$, represented by $x*x$, is 0 follows from the fact that x is in Hochschild degree 0. To determine $Q_1(u)$ (represented by $\overline{u} * \overline{u}$), note that

$$\overline{u} * \overline{u}(x^{2k}) = \overline{u}(\overline{u}(x^{2k})) = \overline{u}(0) = 0$$

and

$$\overline{u} * \overline{u}(x^{2k+1}) = \overline{u}(\overline{u}(x^{2k+1})) = \overline{u}(x^{2k+1}) = x^{2k+1}$$

so $\overline{u} * \overline{u} = \overline{u}$. Hence $Q_1(u) = u$. The same kind of argument shows that $Q_1(v) = 0$. Note, however, that xv is represented by the same element as u, so this gives a demonstration of the fact that $Q_1(xv) = xv$.

To see that $\lambda_1(x, v) = 1$, note that $\lambda_1(x, v)$ is represented by

$$x * \overline{v} - \overline{v} * x = -\overline{v} * x = \overline{v}(x) = 1$$

since $x * v = 0$ and $f * g$ is just the substitution $f(g)$ when $f \in CH^1(A, A)$ and $g \in CH^0(A, A) \cong A$. Similarly, $\lambda_1(x, u)$ is represented by $\overline{u} * x = \overline{u}(x) = x$.

We now show that $\lambda_1(u, t) = t$. Let $\overline{t} \in CH^2(R, R)$ represent t. Since the topological degree of t is $-2 - r(n + 1)$ and its Hochschild degree is 2, as a graded function $R \otimes R \to R$, $\overline{t}$ lowers degree by $r(n + 1)$. Therefore we may write

$$\overline{t}(x^i \otimes x^j) = t_{i,j} x^{i+j-(n+1)}$$

for some coefficients $t_{i,j} \in \mathbb{F}_2$. Write $u_i = i \mod 2$, so that $\overline{u}(x^i) = u_i x^i$. Then $\lambda_1(u, t)$ is represented by $\overline{u} * \overline{t} - \overline{t} * \overline{u}$. So, confusing homology classes and their representatives,

$$\begin{aligned} \lambda_1(u, t)(x^i \otimes x^j) &= u(t(x^i \otimes x^j)) + t(u(x^i) \otimes x^j) + t(x^i \otimes u(x^j)) \\ &= u(t_{i,j} x^{i+j-(n+1)}) + t(u_i x^i \otimes x^j) + t(x^i \otimes u_j x^j) \\ &= t_{i,j}(u_{i+j-(n+1)} + u_i + u_j) x^{i+j-(n+1)} \end{aligned}$$

Then, since n is even,

$$u_{i+j-(n+1)} + u_i + u_j = i + j - (n + 1) + i + j = 1 \mod 2$$

so $\lambda_1(u, t)(x^i \otimes x^j) = t_{i,j} x^{i+j-(n+1)}$; hence $\lambda_1(u, t) = t$.

□

Proof of Lemma 4.6

The proofs of all of these vanishings follow the same pattern: we examine which Hochschild degree the element in question must lie in and demonstrate that nothing in that degree could have the right topological degree.

To see that $Q_1(t) = 0$, observe that $Q_1(t)$ is represented by $\overline{t} * \overline{t} \in CH^3(R, R)$. Hence, if nonzero, $Q_1(t) = x^k tu$ (if n is even) or $Q_1(t) = x^k tv$ (if n is odd) for some choice of k. Assume n is even; then

$$|x^k tu| = kr - 2 - r(n + 1) - 1 = (k - n - 1)r - 3$$

Generally, for any class a, $|Q_1(a)| = 2|a| + 1$, so

$$|Q_1(t)| = 2(-2 - r(n + 1)) + 1 = (-2n - 2)r - 3$$

Therefore (since $r \neq 0$), $k = -n - 1 < 0$ which is false. So $Q_1(t) = 0$. A similar argument holds if n is odd.

$\lambda_1(x, t)$ lies in $HH^1(R, R)$ (as it is represented by $x * t - t * x$). So, if nonzero, it is $x^k u$ or $x^k v$ (again, depending on the parity of n), and

$$|x^k u| = kr - 1, \quad |x^k v| = kr - 1 - r = (k - 1)r - 1$$

For any a and b, $|\lambda_1(a, b)| = |a| + |b| + 1$, so

$$|\lambda_1(x, t)| = r - 2 - r(n + 1) + 1 = -nr - 1$$

If n is even, then we must again have $k = -n < 0$, and so $\lambda_1(x, t) = 0$. If n is odd, $k = 1 - n$. If $n > 1$, then $\lambda_1(x, t) = 0$ for the same reason. But if $n = 1$, the algebra collapses (to the sphere case):

$$HH^*(R,R) = \mathbb{F}_2[x,v]/(x^2)$$

and $v^2 = t$. Then

$$\lambda_1(x,t) = \lambda_1(x,v^2) = \lambda_1(x,v)v + v\lambda_1(x,v) = 0$$

So generally $\lambda_1(x,t) = 0$.

Finally, $\lambda_1(v,t) \in HH^2(R,R)$. So if nonzero, it must be $x^k t$ for some k.

$$|x^k t| = kr - 2 - r(n+1) = (k-n-1)r - 2$$

but

$$|\lambda_1(v,t)| = -1 - r - 2 - r(n+1) + 1 = (-n-2)r - 2$$

so $k = -1$, a contradiction.

□

5. The case of $\mathbb{R}P^n$

In this section, we establish the part of Corollary 2.2 which does not follow immediately from Proposition 2.1. That is, for $n > 1$

$$H_*(L\mathbb{R}P^{n-T\mathbb{R}P^n}) \cong HH^*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n))$$

This is proved for the other manifolds under consideration using the Bousfield-Kan spectral sequence for a cosimplicial space. For $\mathbb{R}P^n$ we will use an Eilenberg-Moore spectral sequence for a fibration; this spectral sequence will be equivalent to the cosimplicial one.

For a space M, let PM be its free path space, $ev : PM \to M \times M$ the evaluation at both endpoints (a fibration), and $\Delta : M \to M \times M$ the diagonal. Then the pullback of ev over Δ is a fibration with LM as the total space, and the projection is evaluation at the basepoint of S^1 (which we will also denote ev):

$$\begin{array}{ccc} LM & \longrightarrow & PM \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev} \\ M & \xrightarrow[\Delta]{} & M \times M \end{array}$$

The fibre of these fibrations is homotopy equivalent to ΩM so, if M is connected and simply connected, there is an Eilenberg-Moore spectral sequence converging strongly to $H^*(LM)$ with E_2 term

$$E_2 = \mathrm{Tor}_{H^*(M\times M)}(H^*(M), H^*(PM))$$

Since PM is equivalent to M (and via this equivalence ev becomes Δ), we may write $E_2 = HH_*(H^*(M), H^*(M))$. The construction of the spectral sequence is such that it converges to $HH_*(C^*(M), C^*(M)) \cong H^*(LM)$. If M is formal, then the spectral sequence collapses, and so

$$H^*(LM) \cong HH_*(H^*(M), H^*(M))$$

One may then follow the arguments of [CJ02] employing Poincaré duality to conclude that

$$H_*(LM^{-TM}) \cong HH^*(H^*(M), H^*(M))$$

If M is not simply connected, the fibre of ev is not connected, so we cannot use the Eilenberg-Moore spectral sequence directly. However, if $\pi_1 M$ is abelian and x is a given point in M, there is a continuous map

$$LM \to \pi_1(M,\, x)$$

given by $f \mapsto [f] \in \pi_1(M,\, ev(f)) \cong \pi_1(M,\, x)$, where the isomorphism is canonical. So we see that

$$LM = \coprod_{g \in \pi_1(M,\, x)} L_g M$$

where $L_g M \subseteq LM$ are the loops freely homotopic to g.

Write $\tilde{M}$ for the universal cover of M and $\tilde{M} \times_{\pi_1 M} \tilde{M}$ for the cover of $M \times M$ corresponding to the diagonal subgroup $\Delta_*(\pi_1 M) \subseteq \pi_1 M \times \pi_1 M$. The map $ev : PM \to M \times M$ lifts to $\tilde{M} \times_{\pi_1 M} \tilde{M}$. Fix a point $y \in \tilde{M} \times_{\pi_1 M} \tilde{M}$ lying over $(x, x) \in M \times M$, and let ev_0 be the lift of ev carrying the constant map at x to y. Then for each $g \in \pi_1 M$ let $ev_g = g \circ ev_0$, where g is thought of as a deck transformation in the right hand side of the equation. Similarly (and homotopically equivalent), let Δ_0 be a lift of Δ carrying x to y, and $\Delta_g = g \circ \Delta_0$.

Now $ev_g : PM \to \tilde{M} \times_{\pi_1 M} \tilde{M}$ is a fibration and the pullback of ev_g along Δ_0 is $L_g M$:

$$\begin{array}{ccc} L_g M & \longrightarrow & PM \\ {\scriptstyle ev}\big\downarrow & & \big\downarrow{\scriptstyle ev_g} \\ M & \xrightarrow[\Delta_0]{} & \tilde{M} \times_{\pi_1 M} \tilde{M} \end{array}$$

The fibre of these fibrations are connected (being the component of ΩM containing g). Moreover, $\pi_1(\tilde{M} \times_{\pi_1 M} \tilde{M}) = \pi_1 M$ is abelian (hence nilpotent) and acts trivially on the homology of the fibre of PM. Therefore by Shipley's general convergence results in [Shi96], the Eilenberg-Moore spectral sequence for this fibration converges strongly to $H^*(L_g M)$. In summary,

Theorem 5.1. *If M is connected and $\pi_1 M$ is abelian, there is a spectral sequence converging strongly to $H^*(LM)$ with E_2-term*

$$E_2 = \bigoplus_{g \in \pi_1 M} \mathrm{Tor}_{H^*(\tilde{M} \times_{\pi_1 M} \tilde{M})}(H^*(M), H^*(M))$$

where the $H^*(\tilde{M} \times_{\pi_1 M} \tilde{M})$-module structure on the first copy of $H^*(M)$ is via Δ_0^*, and via $ev_g^* = \Delta_g^*$ on the second.

To apply this to $M = \mathbb{R}P^n$, we need to study the cohomology of $\tilde{M} \times_{\pi_1 M} \tilde{M} = S^n \times_{\mathbb{Z}/2} S^n$. The (cohomological) Serre spectral sequence for the fibration

$$S^n \to S^n \times_{\mathbb{Z}/2} S^n \to \mathbb{R}P^n$$

has as an E_2 term the algebra $H^*(\mathbb{R}P^n) \otimes H^*(S^n) = \mathbb{F}_2[x, a]/(x^{n+1}, a^2)$. The geometry of the spectral sequence implies that it collapses at E_2, so it only remains to determine a^2.

Lemma 5.2.

(1) *If n is odd, $H^*(S^n \times_{\mathbb{Z}/2} S^n) = \mathbb{F}_2[x, a]/(x^{n+1}, a^2)$, and $\Delta_g^*(x) = x$, $\Delta_g^*(a) = 0$ for both values of g.*

(2) *If n is even, $H^*(S^n \times_{\mathbb{Z}/2} S^n) = \mathbb{F}_2[x,a]/(x^{n+1}, a^2 - ax^n)$, and $\Delta_g^*(x) = x$, $\Delta_1^*(a) = 0$, $\Delta_0^*(a) = x^n$.*

Proof. $S^n \times_{\mathbb{Z}/2} S^n$ is the unit sphere bundle, SE, of the bundle $E = S^n \times_{\mathbb{Z}/2} \mathbb{R}^{n+1}$ over $\mathbb{R}P^n$ (where $\mathbb{Z}/2$ acts diagonally on both factors by -1). Now $E = T\mathbb{R}P^n \oplus 1$. Therefore SE is homeomorphic via the stereographic projection to the fibrewise one-point compactification of $T\mathbb{R}P^n$, which we will denote $\overline{T\mathbb{R}P^n}$. In this description, the two lifts Δ_0 and Δ_1 of the diagonal are the zero-section and ∞-section of $\overline{T\mathbb{R}P^n}$, respectively.

Consequently, this is a cofibration sequence

$$\mathbb{R}P^n \xrightarrow{\Delta_1} S^n \times_{\mathbb{Z}/2} S^n = \overline{T\mathbb{R}P^n} \xrightarrow{\text{collapse}} \mathbb{R}P^{n^{T\mathbb{R}P^n}}$$

We may take a to be the Thom class of $T\mathbb{R}P^n$, so that this sequence realizes

$$H^*(S^n \times_{\mathbb{Z}/2} S^n) = H^*(\mathbb{R}P^n) \oplus H^*(\mathbb{R}P^n)a = H^*(\mathbb{R}P^n) \oplus \tilde{H}^*(\mathbb{R}P^{n^{T\mathbb{R}P^n}})$$

Then $a^2 = Sq^n a = w_n(T\mathbb{R}P^n)a = (n+1)x^n a$. This proves the claims about ring structure.

Regarding the value of Δ_g^*, the fact that it is a lift of the diagonal implies that it always carries x to x. The cofibration sequence above shows that $\Delta_1^*(a) = 0$ for every value of n. Since Δ_0 is the zero-section and a is the Thom class, $\Delta_0^*(a) = w_n(T\mathbb{R}P^n) = (n+1)x^n$.

□

The E_2-term of the spectral sequence of Theorem 5.1, in the case $M = \mathbb{R}P^n$, has two summands corresponding to the two elements of $\mathbb{Z}/2$. We will write them Tor_0^* and Tor_1^*. Also, we continue to write $R = H^*(\mathbb{R}P^n) = \mathbb{F}_2[x]/(x^{n+1})$.

Corollary 5.3.

(1) *If n is odd, $\mathrm{Tor}_g^m = R$ for all m and g.*

(2) *If n is even,*

$$\mathrm{Tor}_0^m = \begin{cases} R & m = 0 \\ \Sigma^{-mn}R/(x^n) & m \text{ odd} \\ \Sigma^{-mn}xR & m > 0, \text{ even} \end{cases}$$

$$\mathrm{Tor}_1^m = \begin{cases} \Sigma^{-mn}R/(x^n) & m \text{ even} \\ \Sigma^{-mn}xR & m \text{ odd} \end{cases}$$

Proof. From Lemma 5.2, if n is odd, a free resolution of R over $S = H^*(S^n \times_{\mathbb{Z}/2} S^n)$ is the periodic resolution

$$0 \longleftarrow R \xleftarrow{\epsilon} S \xleftarrow{a} \Sigma^{-n}S \xleftarrow{a} \Sigma^{-2n}S \xleftarrow{a} \Sigma^{-3n}S \xleftarrow{a} \dots$$

regardless of whether R is an S-module by way of Δ_0^* or Δ_1^*. The first claim follows.

If n is even, a free S-module resolution of R with S-module structure given by Δ_0^* is also periodic, but of period 2:

$$0 \longleftarrow R \xleftarrow{\epsilon} S \xleftarrow{a+x^n} \Sigma^{-n}S \xleftarrow{a} \Sigma^{-2n}S \xleftarrow{a+x^n} \Sigma^{-3n}S \xleftarrow{a} \dots$$

So Tor_0^* is the homology of

$$R \xleftarrow{0} \Sigma^{-n}R \xleftarrow{x^n} \Sigma^{-2n}R \xleftarrow{0} \Sigma^{-3n}R \xleftarrow{x^n} \dots$$

and Tor_1^* is the homology of

$$R \xleftarrow{x^n} \Sigma^{-n}R \xleftarrow{0} \Sigma^{-2n}R \xleftarrow{x^n} \Sigma^{-3n}R \xleftarrow{0} \dots$$

□

We now prove the final piece of Corollary 2.2:

Lemma 5.4.
$$H_*(L\mathbb{R}P^{n-T\mathbb{R}P^n}) \cong HH^*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n))$$

Proof. The covering map $S^n \times_{\mathbb{Z}/2} S^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ induces a map

$$p : \mathrm{Tor}_{C^*(\mathbb{R}P^n \times \mathbb{R}P^n)}(C^*\mathbb{R}P^n, C^*\mathbb{R}P^n) \to \bigoplus_{g \in \mathbb{Z}/2} \mathrm{Tor}_{C^*(S^n \times_{\mathbb{Z}/2} S^n)}(C^*\mathbb{R}P^n, C^*\mathbb{R}P^n)$$

There are Eilenberg-Moore spectral sequences converging to both the domain and codomain of this map. The formality result, Proposition 2.1, implies that the spectral sequence for the domain collapses at E_2, and that the domain is isomorphic to $HH_*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n))$. This may be computed from Proposition 3.1 as the homology of

$$R \xleftarrow{0} \Sigma^{-1}R \xleftarrow{(n+1)x^n} \Sigma^{-n-1}R \xleftarrow{0} \Sigma^{-n-2}R \xleftarrow{(n+1)x^n} \Sigma^{-2n-2}R \cdots$$

The spectral sequence for the codomain has E_2-term $\oplus_{g \in \mathbb{Z}/2}\mathrm{Tor}^*_g$. This spectral sequence converges strongly to $H^*(L\mathbb{R}P^n)$, so the map above can be thought of as a function between $HH_*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n))$ and a subquotient of $\oplus_{g \in \mathbb{Z}/2}\mathrm{Tor}^*_g$. In fact, it is an honest isomorphism identifying Tor^{2k}_0 with HH_{4k}, Tor^{2k-1}_0 with HH_{4k-1}, Tor^{2k}_1 with HH_{4k+1}, and Tor^{2k+1}_1 with HH_{4k+2}. This can be seen by comparing terms from Corollary 5.3 with terms from the sequence above.

Consequently, the spectral sequence computing $H^*(L\mathbb{R}P^n)$ (i.e., the codomain of p) must necessarily collapse. Therefore p is an isomorphism

$$p : HH_*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n)) \cong H^*(L\mathbb{R}P^n)$$

Using this fact, one may proceed as in [CJ02] to show through Poincaré duality that

$$H_*(L\mathbb{R}P^{n-T\mathbb{R}P^n}) \cong HH^*(H^*(\mathbb{R}P^n), H^*(\mathbb{R}P^n))$$

□

References

[Abb03] H. Abbaspour, *On string topology of three manifolds*, preprint: math.GT/0310112 (2003).

[AK56] S. Araki and T. Kudo, *Topology of H_n-spaces and H-squaring operations*, Mem. Fac. Sci. Kyusyu Univ. Ser. A, 1956, pp. 85–120.

[Bro60] W. Browder, *Homology operations and loop spaces*, Ill. J. Math. **4** (1960), 347–357.

[CJ02] R. L. Cohen and J. D. S. Jones, *A homotopy theoretic realization of string topology*, Mathematische Annalen **324 no. 4** (2002), 773–798.

[CJY02] R. L. Cohen, J. D. S. Jones, and J. Yan, *The loop homology algebra of spheres and projective spaces*, preprint: math.AT/0210353 (2002).

[CLM76] F. R. Cohen, T. J. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics 533, Springer Verlag, Berlin, 1976.

[Coh04] R. L. Cohen, *Multiplicative properties of Atiyah duality*, preprint (2004).

[CS01] M. Chas and D. Sullivan, *String topology*, preprint: math.GT/9911159 (2001).

[DL62] E. Dyer and R. K. Lashof, *Homology of iterated loop spaces*, Amer. J. Math. (1962), 35–88.

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Mathematics **78** (1963), 267–288.

[KS00] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conference Moshe Flato 1999, Math. Phys. Stud., 22, vol. II, 2000, pp. 255–307.

[Lod92] J.-L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, Berlin, 1992.

[May67] J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand, Princeton, 1967.

[May70] ______, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications, Lecture Notes in Math, no. 168, Springer-Verlag, 1970, pp. 153–231.

[MS02] J. E. McClure and J. H. Smith, *A solution of Deligne's Hochschild cohomology conjecture*, Recent progress in homotopy theory (Baltimore, MD, 2000) (Providence, RI), Contemporary Mathematics, no. 293, Amer. Math. Soc, 2002, pp. 153–193.

[Shi96] B. Shipley, *Convergence of the homology spectral sequence of a cosimplicial space*, American Journal of Mathematics **118** (1996), 179–208.

[Tam98a] D. E. Tamarkin, *Another proof of M. Kontsevich formality theorem*, preprint: math.QA/9803025 (1998).

[Tam98b] ______, *Formality of chain operad of small squares*, preprint: math.QA/9809164 (1998).

[Vor00] A. A. Voronov, *Homotopy Gerstenhaber algebras*, Conference Moshe Flato 1999, Math. Phys. Stud., 22, vol. II, 2000, pp. 307–331.