Unit Distance Graphs With Ambiguous Chromatic Number

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Outline

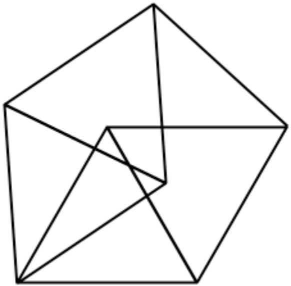
Reference: *Unit distance graphs with ambiguous chromatic number,* Electron. J. of Combin. 16 (2009).

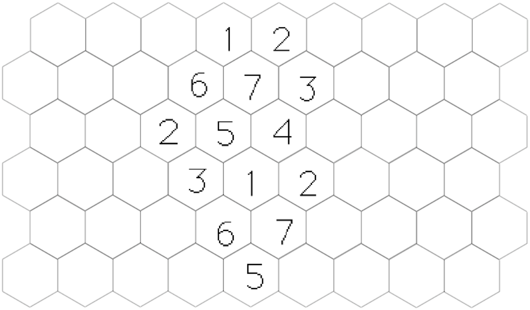
Chromatic Number of the Plane

Problem

What is the minimum number $\chi(\mathbb{R}^2)$ of colours required to colour $\mathbb{R}^2$ so that no two points distance 1 apart receive the same colour?

It is known that $4 \leq \chi(\mathbb{R}^2) \leq 7$.





Previous Results

Theorem [Falconer, 1981]

If we demand that the colour sets be Lebesgue measurable then $n + 3 \leq \chi_m(\mathbb{R}^n)$.

Theorem [Woodall, 1973]

The rational plane can be 2-coloured, hence $\chi(\mathbb{Q}^2) = 2$.

Set Theory

- Completely general colourings can be done in ZFC.
- Restriction to measurable colourings is compatible with restricting the Axiom of Choice to countable collections and imposing
  LM : *All sets in* $\mathbb{R}^n$ *are Lebesgue measurable.*

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Previous ambiguous examples:

- [Székely, 1984] describes a unit distance graph on $\mathbb{S}^1$ with $\chi = 2$ but $\chi_m = 3$.
- [Soifer and Shelah, 2004] describe non unit distance graphs on $\mathbb{R}^2$ and later $\mathbb{R}^n$.

New examples

For any field $\mathbb{Q} \subset K \subset \mathbb{R}$ the graph T_{K^n} is defined by translating the unit distance graph K^n everywhere in $\mathbb{R}^n$.

- $V = \mathbb{R}^n$
- $E = \{\{p_1, p_2\} : p_1 - p_2 \text{ is a unit vector in } K^n\}$

ZFC Colourings

We can show that $\chi(T_{K^n}) = \chi(K^n)$.

- Each coset in $\mathbb{R}^n/K^n$ is disconnected from the others, so can be coloured like K^n.
- For K countable we have to choose uncountably many representatives, one for each coset.

Measurable Colourings

The graphs T_{K^n} have two useful properties:

- The neighbourhood of each vertex is dense in $\mathbb{S}^{n-1}$.
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Lemma

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Lemma

If S is Lebesgue measurable and $\tilde{S}$ realises distance 1, then T_{K^n} has an edge with both vertices in S.

- Choose a unit vector v in K^n very close to $p_1 - p_2$.
- There are neighbourhoods of p_1 and p_2 on which S has large measure.
- Consider translates of v in these neighbourhoods.

Bounds

Theorem

$\chi_m(T_{K^n})$ is at least $\chi(\mathbb{R}^n)$.

- Consider a colouring by $\chi(\mathbb{R}^n) - 1$ measurable sets $\{S_i\}$.
- $\chi(\mathbb{R}^n)$ is attained on a finite unit distance graph.
- This graph can be placed with each vertex in $\tilde{S}_i$ for some i.
- By the lemma T_{K^n} has a monochromatic edge.

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Theorem

It is possible to adapt Falconer's proof to show that $n + 3 \leq \chi_m(T_{K^n})$.

Ambiguity

Corollary

Suppose $\chi(K^n) < n+3$ or $\chi(\mathbb{R}^n)$, then T_{K^n} has ambiguous chromatic number.

- Our original example is $T_{\mathbb{Q}^2}$. Here $\chi = 2$ and $\chi_m \geq 5$.
- For $T_{\mathbb{Q}^3}$ we have $\chi = 2$ and $\chi_m \geq 6$ [Benda and Perles, 2000].
- For $T_{\mathbb{Q}^4}$ we have $\chi = 4$ and $\chi_m \geq 7$ [Benda and Perles, 2000].
- For $K = \mathbb{Q}[\sqrt{3}], n = 2$ we have $\chi = 3$ while $\chi_m \geq 5$ [Fischer, 1990].
- ...

References

Benda, M. and Perles, M. (2000).
Colorings of metric spaces.
Geombinatorics, 9(3):113–126.

Falconer, K. J. (1981).
The realization of distances in measurable subsets covering $\mathbb{R}^n$.
J. Combin. Theory Ser. A, 31(2):184–189.

Fischer, K. G. (1990).
Additive K-colorable extensions of the rational plane.
Discrete Math., 82(2):181–195.

Soifer, A. and Shelah, S. (2004).
Axiom of choice and chromatic number: examples on the plane.
J. Combin. Theory Ser. A, 105(2):359–364.

Székely, L. A. (1984).
Measurable chromatic number of geometric graphs and sets without some distances in Euclidean space.
Combinatorica, 4(2-3):213–218.

Woodall, D. R. (1973).
Distances realized by sets covering the plane.
J. Combin. Theory Ser. A, 14:187–200.