Finding k-angulations on point sets.

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A general type of question

Q: Given a point set P and a graph G, is it possible to draw G on P?

- ▷ All point sets are in general position in the plane.
- ▷ All graphs are drawn with straight edges and no crossings.

A little more precisely...

- A plane graph G is a planar graph together with a drawing in the plane.

- A point set P admits a plane graph G if there is a drawing of G with vertex set P that has the same facial structure, including the same outer face.

Q: Given a class of plane graphs $\mathcal{C}$, which point sets admit a graph from $\mathcal{C}$?

Previous Work

- ▷ [Gritzmann, Mohar, Pach & Pollack '91]
  Every point set P admits every outer-planar graph with $|P|$ vertices.

- ▷ [Bose & Toussaint '97]
  It is always possible to k-angulate $\mathrm{conv}(P)$ if permitted to add up to $k-3$ points.

K-angulations

A **k-angulation** is a 2-connected plane graph in which every internal face is a k-gon. $(k \geq 3)$

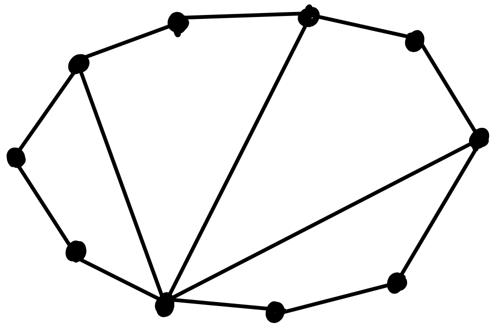
▷ 2-connectedness ensures every face is bounded by a cycle and every edge is in two faces.

$\boxed{\text{For } |P| \geq 3k^2 \text{, we characterise the point sets } P \text{ which admit a } k\text{-angulation.}}$

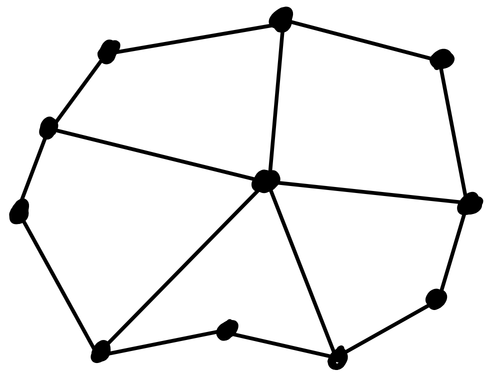
Theorem: Let $n \geq 3k^2$ and $j \equiv_{k-2} 2-n$ with $0 \leq j \leq k-3$. A set of n points admits a k-angulation if and only if it has at least j interior points.

Examples $k=3$: no restriction

$k=4$, n even:



$k=4$, n odd:
need 1 interior point.



Theorem: Let $n \geq 3k^2$ and $j \equiv_{k-2} 2-n$ with $0 \leq j \leq k-3$. A set of n points admits a k-angulation if and only if it has at least j interior points.

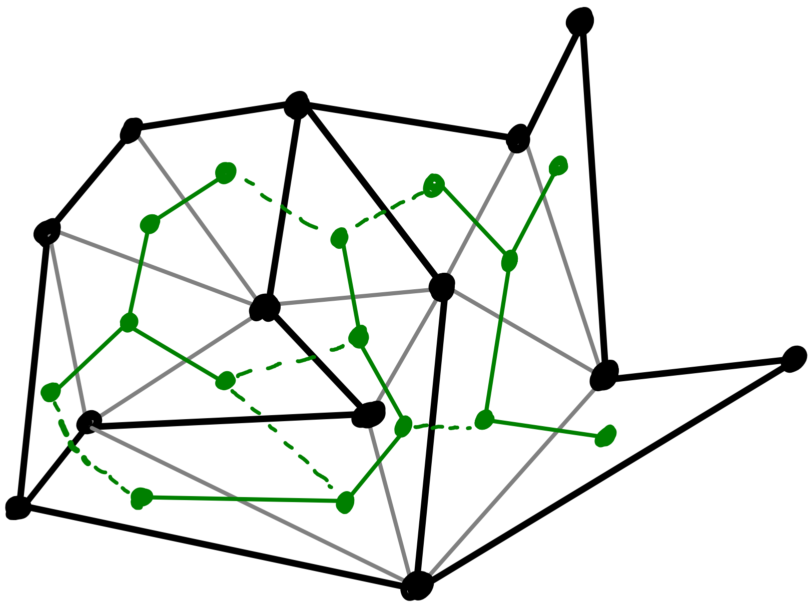
Proof ($\Rightarrow$):

- ▷ Suppose P admits a k-angulation G.
- ▷ Let r be the size of the outer face of G.
- ▷ Vertices of $\operatorname{conv}(P)$ must lie in the outer face, so there are at least $n-r$ interior points.
- ▷ Edges: $2e = k(f-1) + r$ Euler: $n - e + f = 2$.

Combine:
$$n - r = n - 2(n+f-2) + k(f-1)$$
$$= 2 - n + (f-1)(k-2) \equiv_{k-2} 2-n. \quad \square$$

More definitions

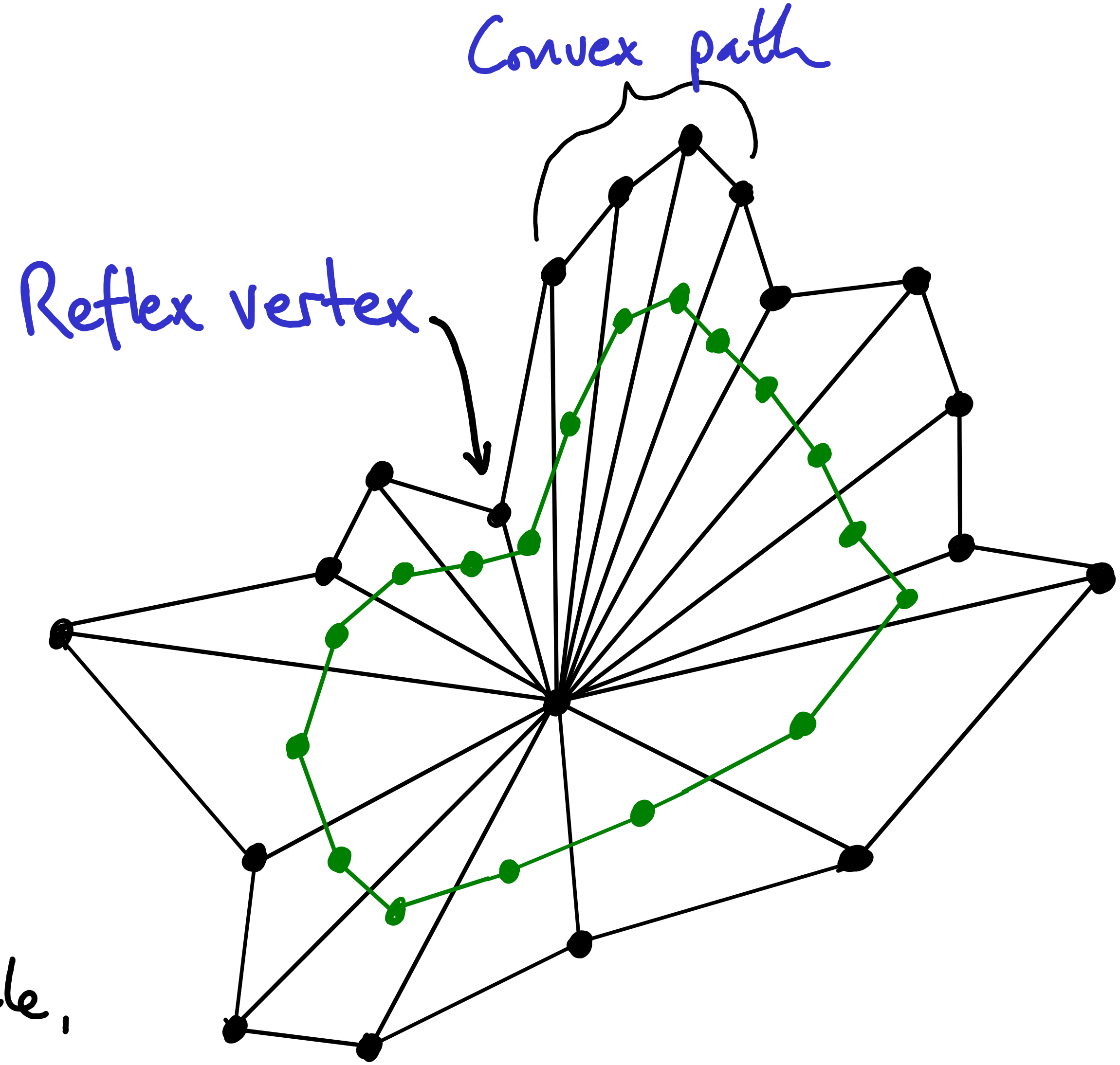
- The weak dual graph of G is obtained by deleting the external face vertex in the planar dual.
- We seek a triangulation which breaks into k-gons. The weak dual must decompose into induced trees of order $k-2$. We call these blocks.



Interior star triangulation

Outer cycle C

Dual cycle Z



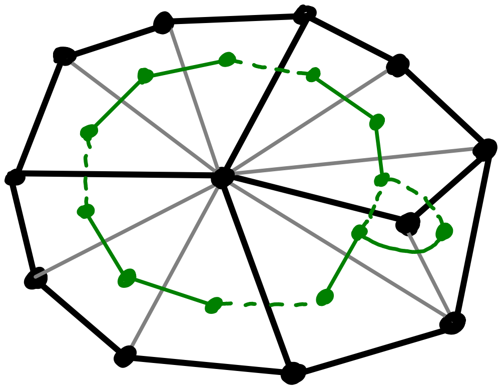
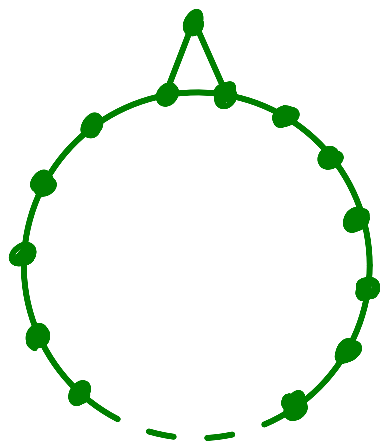
General idea for our construction:

- Begin with i.s.t.,
- add triangles on the outside,
- retriangulate a little,
- decompose into blocks.

Theorem: Let $n \geq 3k^2$ and $j \equiv_{k-2} 2-n$ with $0 \leq j \leq k-3$. A set of n points admits a k-angulation if and only if it has at least j interior points.

Proof ($\Leftarrow$): $\underline{j=0}$: $n-2 \equiv 0$ triangles in an outer planar triangulation.

$\underline{j=1}$: Interior star triangulation has $n-1 \equiv 0$ triangles. Break cycle Z into paths.

$\underline{j=2}$:  Add one triangle. Dual looks like: 

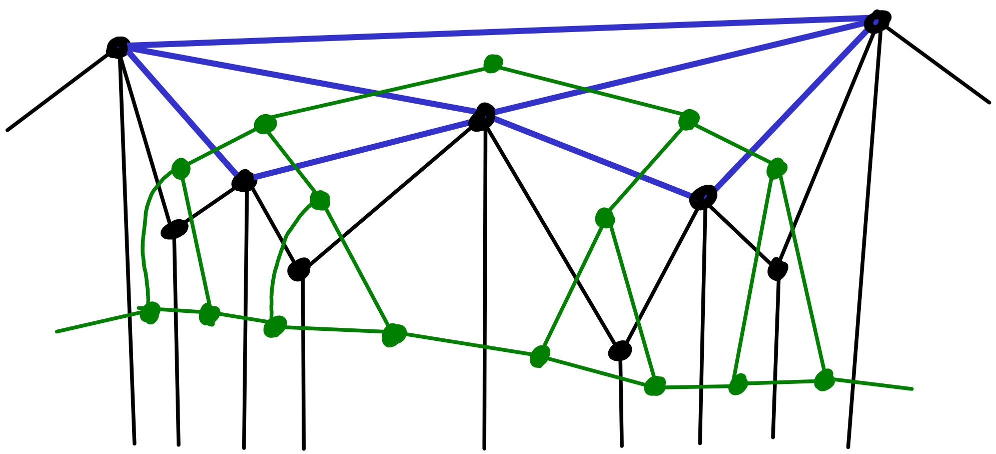
$2 \le j \le k-3$: ▷ Begin with interior star triangulation and add $m := j-1$ triangles to the outside.

▷ Use a selection algorithm which ensures that:

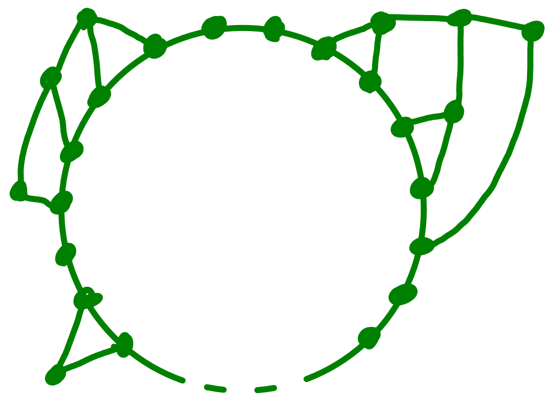
▷ there are no reflex vertices between the triangles,

▷ the triangles are as close to Z as possible in the dual.

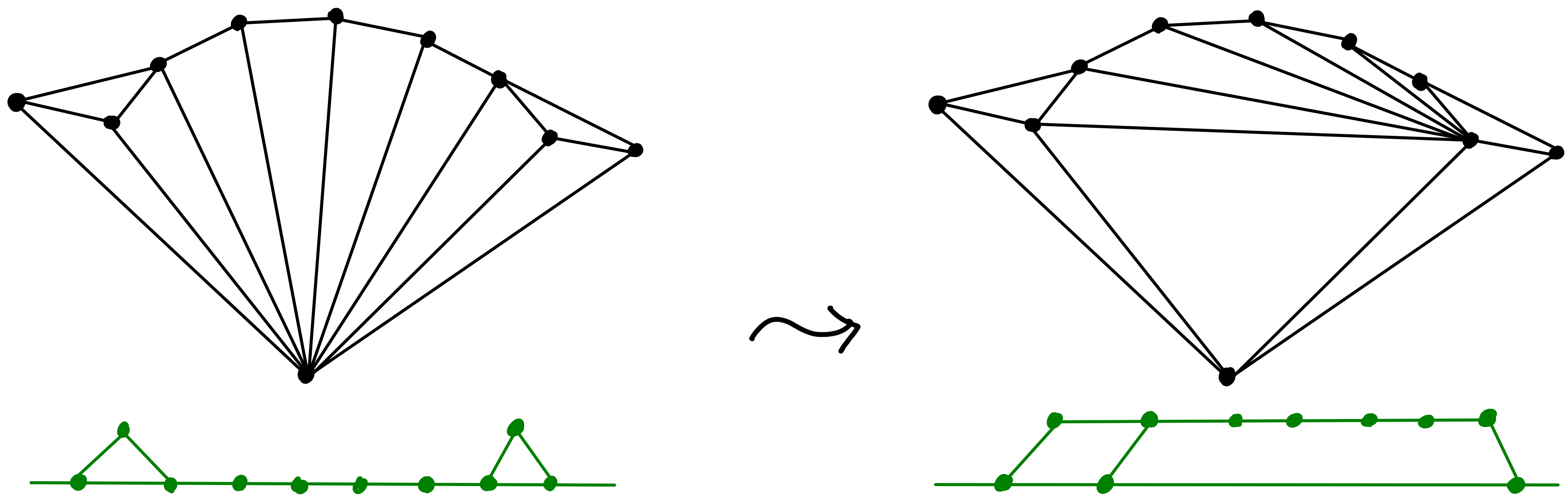
Examples



▷ Dual becomes cycle + trees:

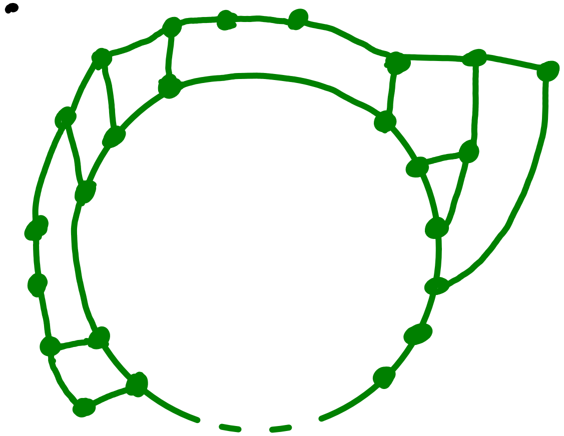


Next, build pontoons:

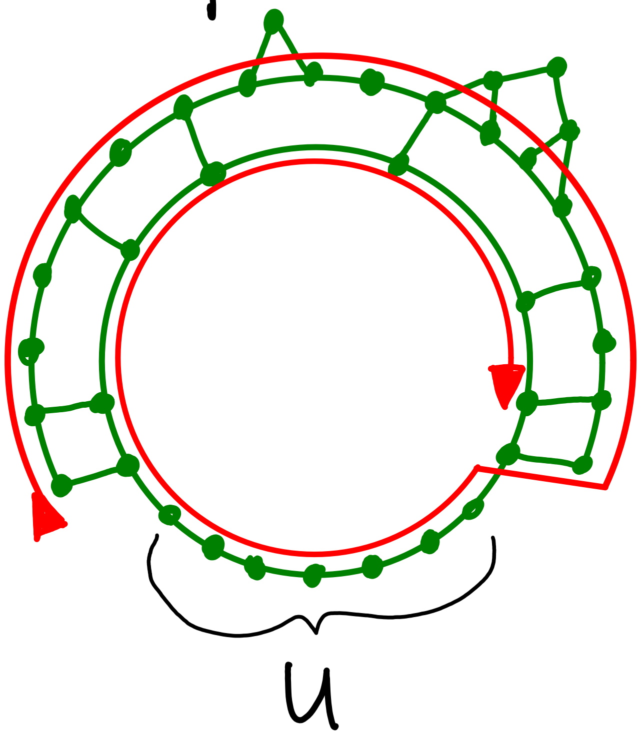


▷ Build pontoons over each convex path between added triangles except one.

▷ New dual looks like:



▷ Decompose the dual along a 'spiral path'

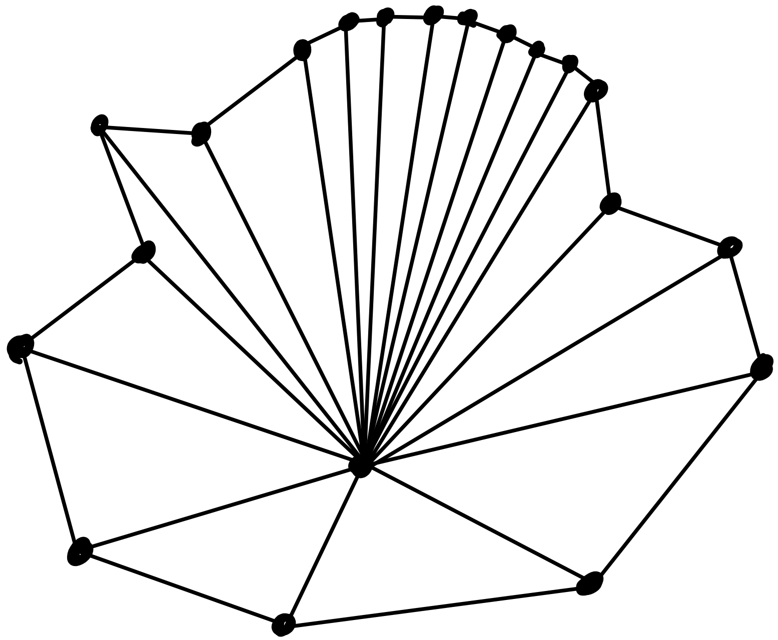


▷ If the choice of added triangles and pontoons was good, then U is large. (Thus blocks are induced trees.)

▷ Added trees may cause problems. Avoid these with a 'launch pad'

▷ $n \geq 3k^2$ ensures large U and space for launch pad.

Q: Where's the problem? Hint: where can you build pontoons?



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