COLOURING THE PLANE

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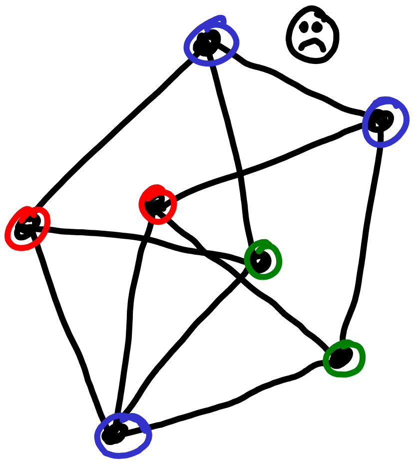
Outline

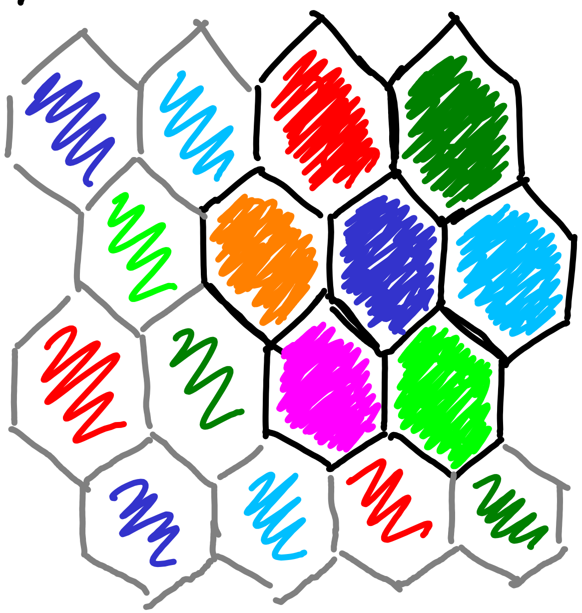
- The problem and its history.
- Variations
- Measurable chromatic number & 'ambiguity'.
- Coordinate fields

The Chromatic Number of the Plane

Q: What is the least number of colours required to colour the plane so that points distance 1 apart receive different colours?

$$4 \le \chi \le 7$$





$d = 1 - \varepsilon$

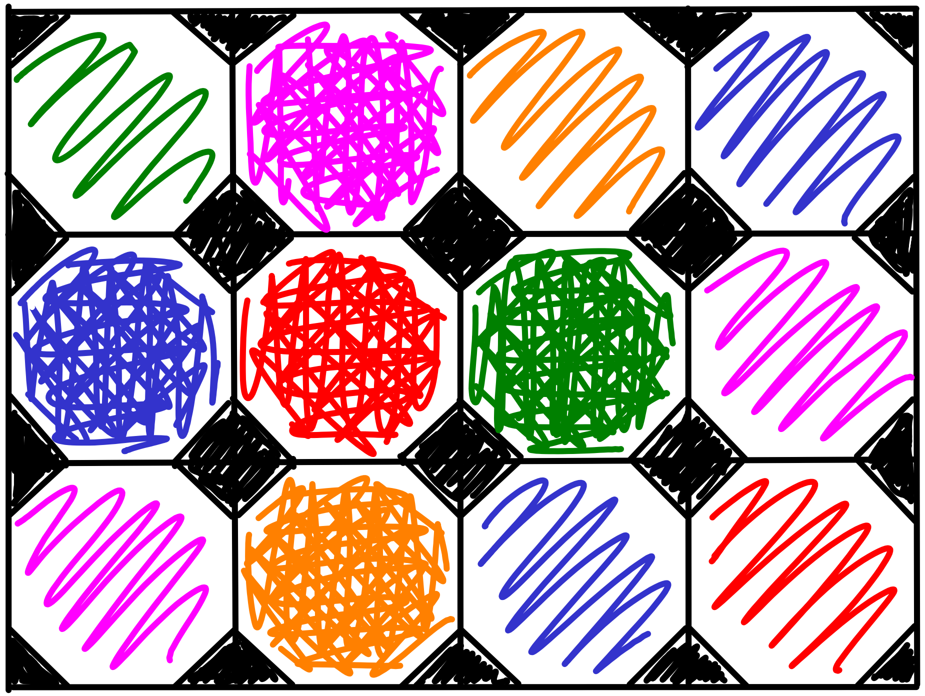
HISTORY

- After much investigation, Soifer attributes the problem to Edward Nelson (1950)
- Martin Gardner made it famous in his 'Mathematical Games' column in *Scientific American* (1960)
- Paul Erdős was a fan and mentioned the problem often. He believed $\chi > 4$.
- $\chi \geq 4$ is due to Nelson, $\chi \leq 7$ to John Isbell ('50)
- For a comprehensive treatment of the problem and its history see Soifer's *Mathematical Coloring Book* (2009)

VARIATIONS

- Restricted Colourings:
  - Regions bounded by Jordan Curves - $\chi \geq 6$ (Woodall '73, corrected by Townsend '81, '05)
  - Lattice based tilings - $\chi = 7$ (Coulson '02)
  - 'Nice', <u>closed</u> regions bounded by Jordan curves - $\chi = 7$ (Thomassen '99)
  - Measurable sets - $\chi \geq 5$ (Falconer '81)
- More excluded distances
- Near colourings, different distance for each colour, etc.
- Other spaces: $\mathbb{R}^n$, S^n, any metric space ...

Bathroom Tiling (see staff toilets)



Soifer's colouring of type $(1, 1, 1, 1, 1, \frac{1}{\sqrt{5}})$.

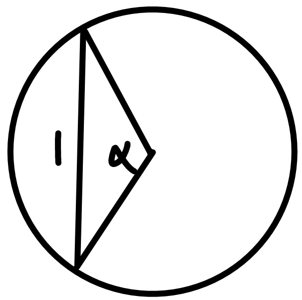
AMBIGUITY

- Maybe $\chi = 4$ while $\chi_{meas} \geq 5$ (!?)
- Can be seen as 'dependence on set theory'

  ZFC vs ZF + DC + LM

- First example (Székely '84)

  $\chi = 2 \quad \chi_m = 3$



$\alpha = \beta\pi$

$\beta \notin \mathbb{Q}$

- More examples (not distance graphs) (Soifer, Shelah '04)
- More: T_{K^n} := all translates of unit dist graph on K^n, $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ (P. '09)

$\underline{T_{K^n}}$: $V = \mathbb{R}^n$

$E = \{\{p_1, p_2\} : p_1 - p_2 \text{ unit vector in } K^n\}$

ZFC Colourings

Proposition: $\chi(T_{K^n}) = \chi(K^n)$

- Each coset in $\mathbb{R}^n / K^n$ can be coloured independently.
- If K is countable we need full AC to choose representatives.

Measurable Colourings

T_{K^n} has two useful properties:

- Neighbourhood of a vertex is dense in S^{n-1}.
- E is invariant under any translation.

Def^n: $S \subseteq \mathbb{R}^n$ measurable. The points where S has Lebesgue density 1 are the **essential part** of S, denoted $\tilde{S}$.

Lemma: If $\tilde{S}$ realises distance 1 then T_{K^n} has an edge with both ends in S.

- Choose unit vector $v \in K^n$ close to $p_1 - p_2$
- $\exists$ n'hoods of p_1, p_2 on which S has large measure.
- Consider translates of v in these n'hoods.

Theorem: $\chi_m(T_{K^n})$ is at least $\chi(\mathbb{R}^n)$.

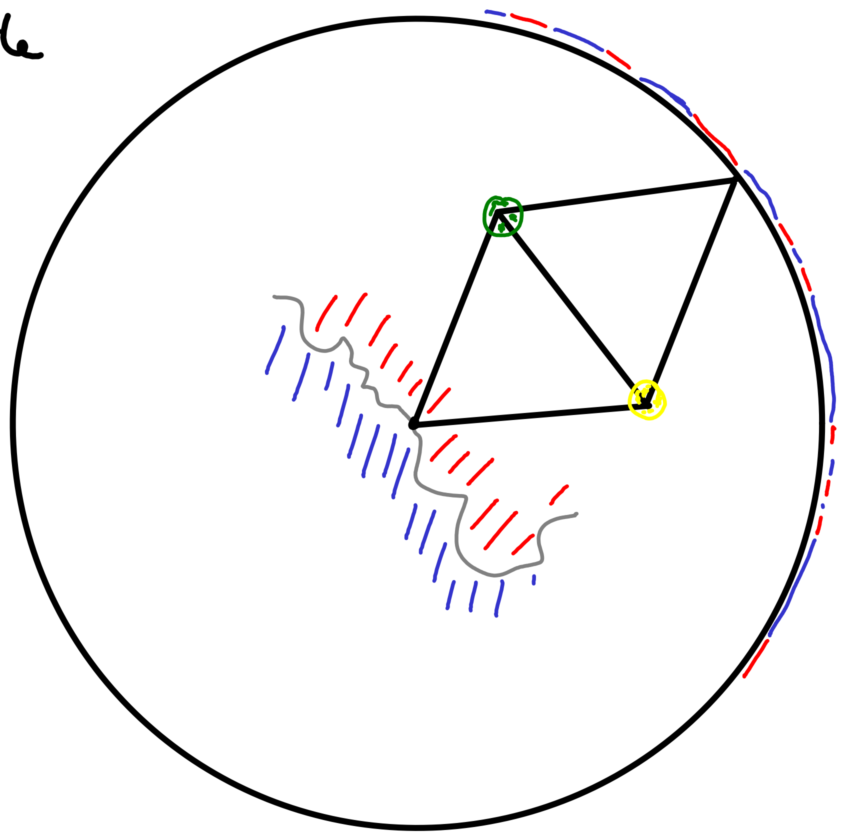
- Consider a colouring with fewer measurable sets.
- $\chi(\mathbb{R}^n)$ is attained on some finite subgraph.
- Can place this graph with each vertex in the essential part of some colour.
- Applying the Lemma, T_{K^n} has a monochromatic edge.

Theorem: $\chi_m(T_{K^n}) \geqslant n+3$.

- This can be shown by adapting Falconer's ('81) proof that $\chi_m(\mathbb{R}^n) \geqslant n+3$.

Idea of Falconer's Proof:

- Suppose we have a measurable $(n+2)$-colouring.
- Almost all of the sphere is in $\tilde{S} \cup \tilde{S}$.
- Because of the radius $\tilde{S}$ (w.l.o.g.) must realise 1.
- But this implies that S realises 1. ↯



Ambiguous Cases

If $\chi(K^n) < n+3$ or $\chi(\mathbb{R}^n)$ then T_{K^n} has $\chi \neq \chi_m$.

K	n	χ	χ_m	
$\mathbb{Q}$	2	2	≥ 5	(Woodall '73)
$\mathbb{Q}$	3	2	≥ 6	(Benda & Perles 2000)
$\mathbb{Q}$	4	4	≥ 7	
$\mathbb{Q}[\sqrt{a}]$ $a \equiv_4 1,2$	2	2	≥ 5	(Johnson '87)
$a \equiv_3 0,1$	2	$3 \geq$	≥ 5	(Fischer '90)
$a \equiv_8 3$	2	$4 \geq$	≥ 5	

How to 2-colour $\mathbb{Q}^2$ (Woodall '73)

- In a reduced Pythagorean triple $a^2+b^2=c^2$, c is odd.
- $\left(\frac{p}{q}, \frac{r}{s}\right)$ unit vector $\Rightarrow (ps)^2+(rq)^2=(qs)^2$ so q, s and one of p or r are odd.
- Define $(a,b)\sim(c,d) \Leftrightarrow$ $a-c$, $b-d$ have odd denominators.
- So points at dist 1 are equivalent. We only need to colour the class containing 0.
- Colouring: $\left(\frac{e}{o}, \frac{e}{o}\right)$ $\left(\frac{o}{o}, \frac{o}{o}\right)$ $\left(\frac{e}{o}, \frac{o}{o}\right)$ $\left(\frac{o}{o}, \frac{e}{o}\right)$
- e.g. $\left(\frac{e}{o}, \frac{e}{o}\right) - \left(\frac{o}{o}, \frac{o}{o}\right) = \left(\frac{e-o}{o}, \frac{e-o}{o}\right) = \left(\frac{o}{o}, \frac{o}{o}\right) \Rightarrow$ not unit length!

  etc.

Realisations of unit distance graphs

An algebraic approach to colouring the plane leads us to consider:

Q: Given a unit distance graph, what is the 'smallest' field in which it can be constructed?

- Benda & Perles (2000): $\mathbb{A}$ is always sufficient.
- Probably a very hard question in general.
- Guess: Constructible Numbers?
  - These are numbers lying in a finite tower of quadratic extensions beginning with $\mathbb{Q}$.
  - Do they at least suffice?

Further investigation (questions & ideas).

- Can we 4-colour all $\mathbb{Q}[\sqrt{n}]$ planes?
- If we can 4-colour K^2 can we do $K[\sqrt{a}]^2$?
  i.e. proof by induction.
- Generalise algebraic techniques already used in known K^2 colourings.

Thanks!