Colouring Great Circle Arrangements

Michael Payne
University of Melbourne

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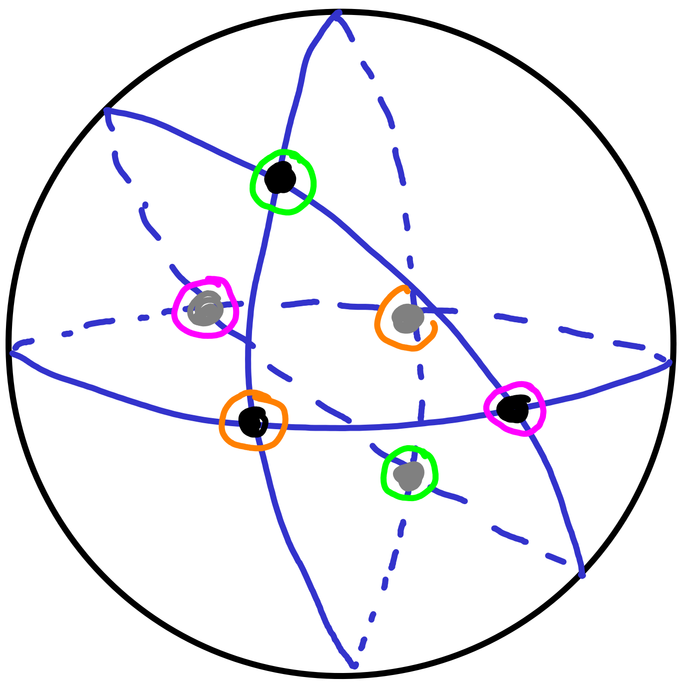
Outline

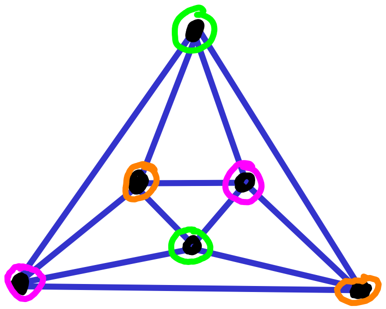
- ▷ Introduce arrangements & conjecture.
- ▷ Why it's interesting / difficult.
- ▷ Approaches & partial results.

The Conjecture

Consider the graph arising from a set of great circles on the sphere. Assume no 3 cross at a point (simple).

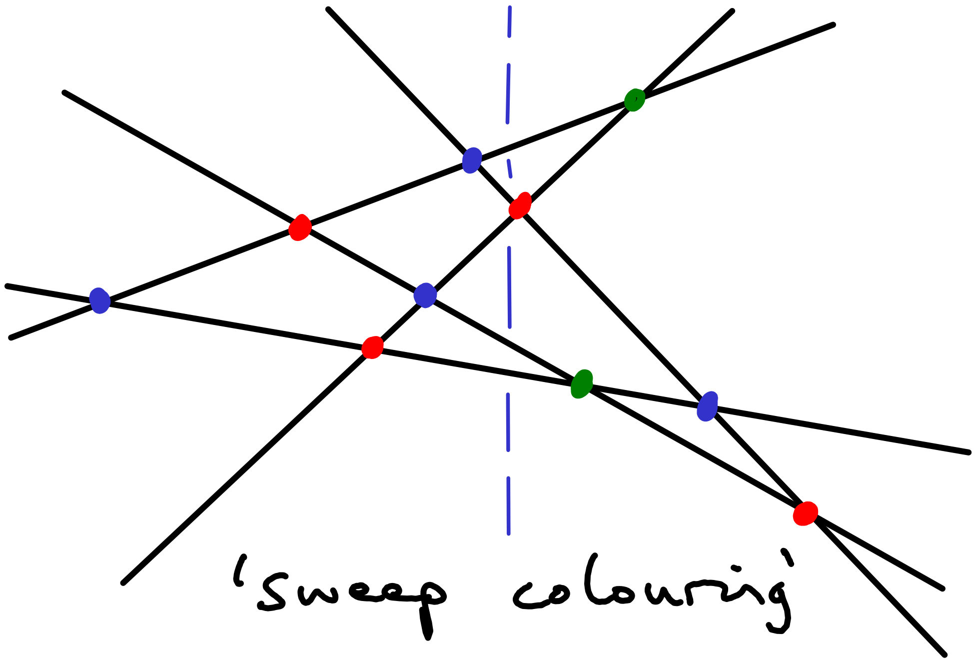
Conjecture: [Felsner et al. '06]
Such graphs are always 3-colourable.



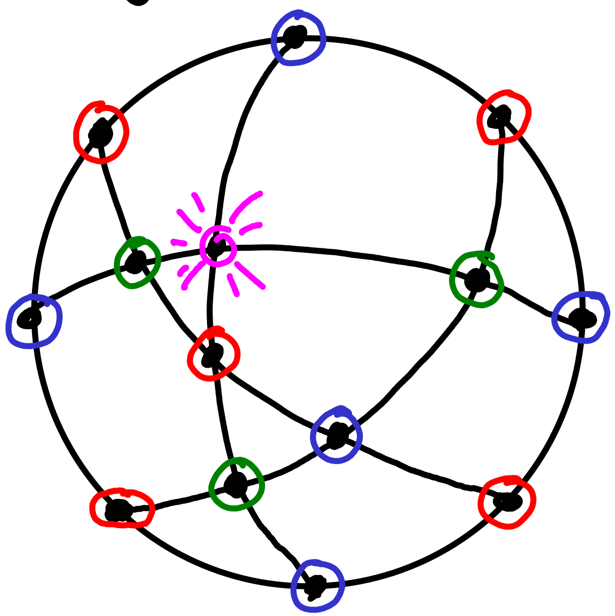


ARRANGEMENTS

- There are various line arrangements associated with a collection of hyperplanes in $\mathbb{R}^3$.
  - Great circle arrangements $(\cap S^2)$
  - Euclidean arrangements: Intersect with affine plane. These are 3-colourable...
  - Projective arrangements: Identify opposite points of a circle arrangement. These may need 4 colours...



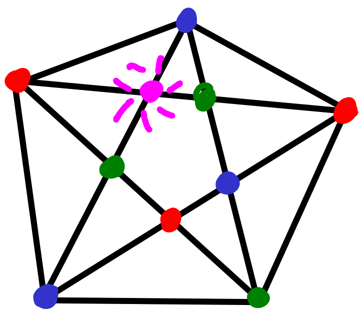
'sweep colouring'

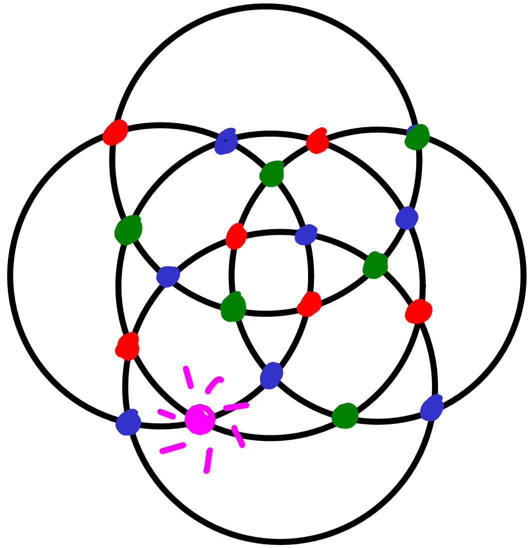


The conjecture can't be weakened:

4 colours are sometimes needed for:

- 'Projective' colourings
- Simple circle graphs
- Planar, 4-regular, 4-connected graphs with 2-hamiltonian-cycle decomposition.
- Non-simple great circle (or line) arrangements.





Koester's 5-circle graph (1984).

Combinatorial Representations

However, we probably only need the combinatorial structure:

- ▷ Any two circles intersect twice.
- ▷ Any third circle separates these intersection points.

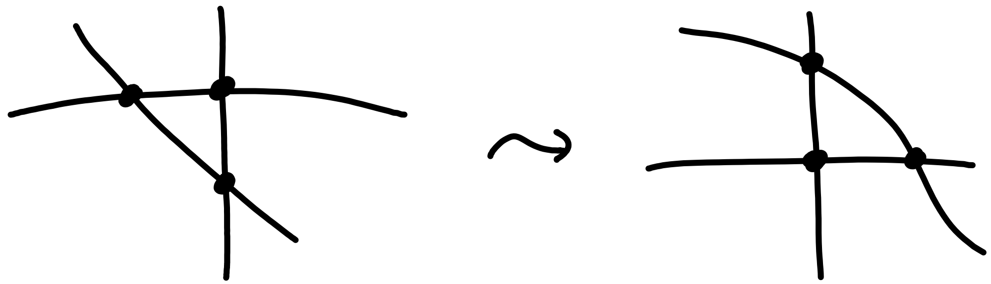
Arrangements of circles which satisfy these axioms are known as 'pseudo-circle arrangements'. These correspond to 'oriented matroids of rank 3'.

There are many combinatorial characterisations of pseudo-circle graphs.

Global Interdependence

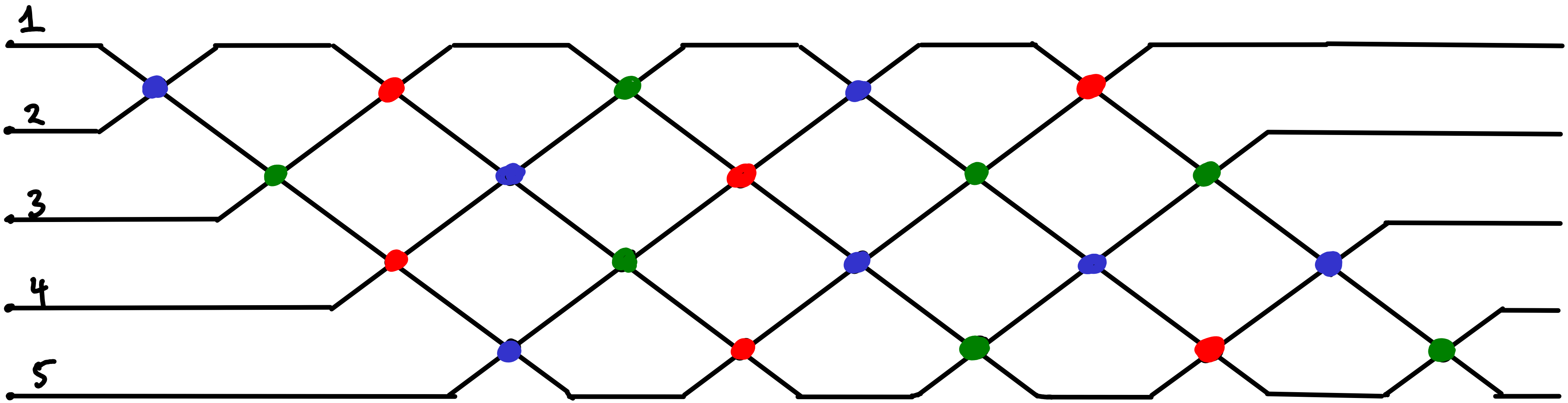
'Local' colouring methods don't seem to work:

- ▷ induction on # circles
- ▷ moving circles around



It seems that if 3-colourings exist then "local changes have global consequences."

The 'canonical arrangement'



- 2 n-gons, 2n triangles, lots of squares
- easy to 3-colour
- easy intersection order

Combinatorial Nullstellensatz.

Defn: An 'Eulerian subgraph' of a graph has even degree at every vertex.

We hoped to apply a powerful theorem of Alon & Tarsi, roughly:

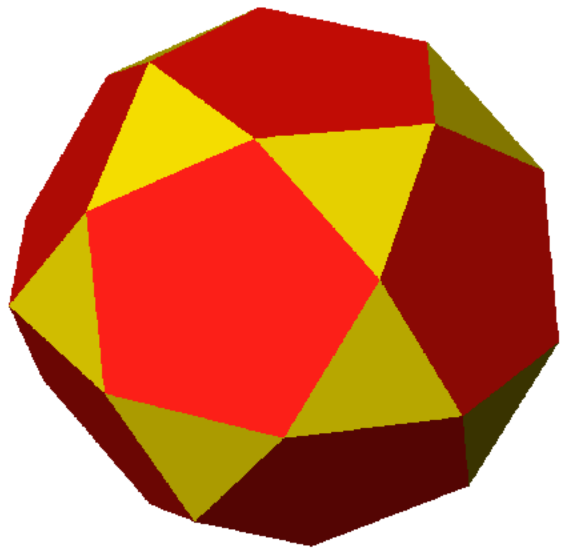
Thm: If a 4-regular graph has a differing number of even vs odd Eulerian subgraphs then it is 3-colourable.

Actually, 3-choosable (list colouring).

Computer Results

We enumerated the Eulerian subgraphs for some small arrangements:

Graph	Even	Odd	Diff
C_2	6	0	6
C_3	22	16	6
C_4	312	288	24
C_5	10,896	10,836	60
C_6	976,922	972,776	4,146
I_6	1,364,348	1,364,288	60

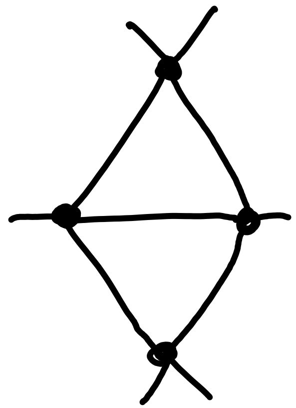


Icosadodecahedron

Counterexamples?

We think that if there are counterexamples they should have many triangles and odd faces (like icosadodecahedron).

▷ An edge can only be in one triangle (for more than 3 circles).
So max triangles $= \frac{2}{3}n(n-1)$.



▷ Are there many 'max-triangle' circle graphs?

Thanks!

Reference

FELSNER ET AL., Hamiltonicity and colourings of arrangement graphs, Discrete Applied Maths, 154 (2006) pp 2470–2483.