On the connectivity of Visibility Graphs

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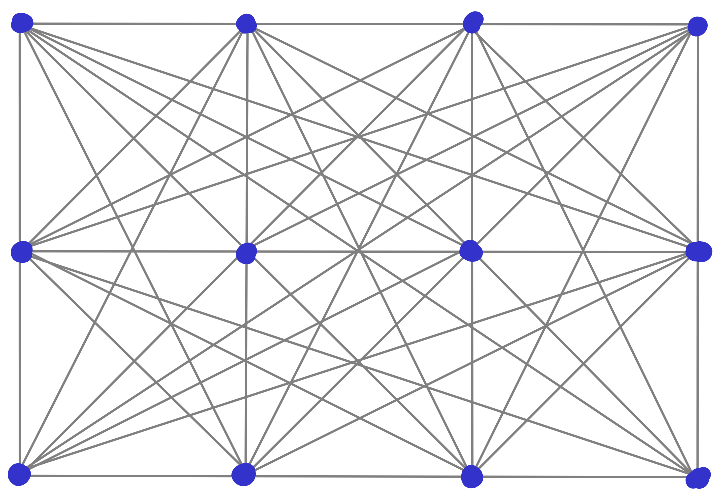
Def^{n} The visibility graph $V(P)$ of a finite set $P \subseteq \mathbb{R}^2$ has P as its vertices and two points are adjacent if they 'see' each other.

i.e. no other point of P lies on the line segment between them.



- Visibility graphs are useful: eg. Szekely's proof of Szemerédi–Trotter Theorem. 'Apply crossing lemma to vis. graph'.

- Clique number vs. chromatic number.

  3 $\Rightarrow$ 3 [Kára, Pór, Wood]

  6 $\rightarrow$ unbounded. [Pfender]

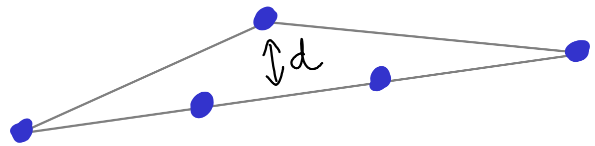
- Big line – big clique conjecture: [K, P, W]

  Every sufficiently large point set in $\mathbb{R}^2$ contains either ℓ collinear points or a clique of size k.

  (True for $k \leq 5$ & $k=6$, $\ell=3$.)

Edge Connectivity

- Visibility graphs have diameter ≤ 2.

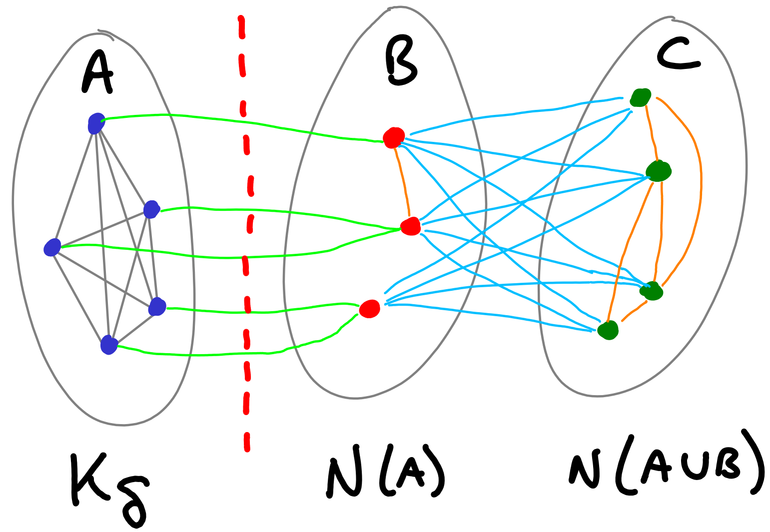


Minimise d.
[K,P,W]

- Diameter 2 graphs have $\lambda = \delta$ [Plesník,'75]

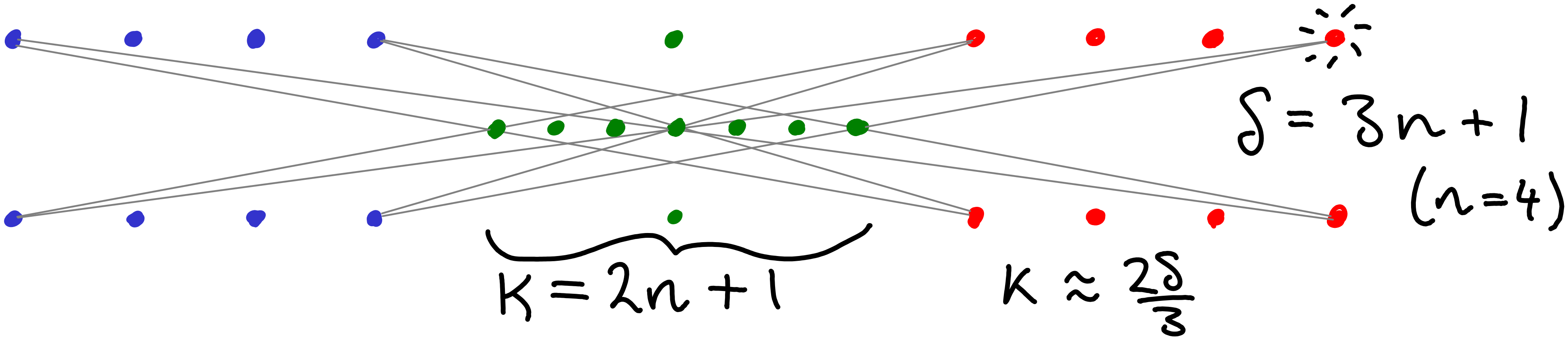
- In visibility graphs, edge cuts of size δ only appear around a vertex.

- Diameter 2 graphs with δ-cuts not around a vertex look like this:



Vertex Connectivity

▷ There are visibility graphs with $\kappa < \delta$.

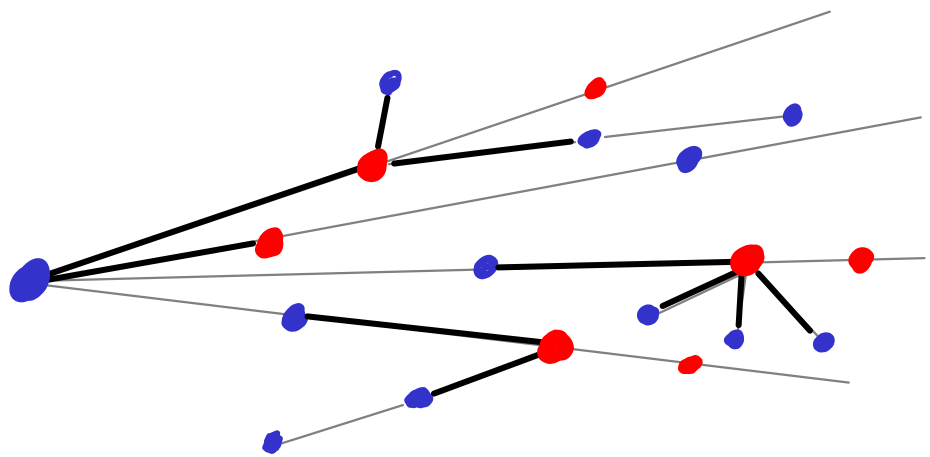


Def^n: The bivisibility graph $B(A,B)$ of two disjoint point sets A and B is the bipartite subgraph of $V(A \cup B)$ induced by A and B.

▷ **Thm:** Let A and B be disjoint sets such that $A \cup B$ has at most ℓ points on a line. Then $B(A,B)$ has a plane subgraph with at least $\frac{n-1}{\ell-1}$ edges. $(n = |A \cup B|)$

▷ **Cor:** If $P \subseteq \mathbb{R}^2$ with $|P| = n$ and at most ℓ points on a line then $V(P)$ is $\frac{n-1}{\ell-1}$ - connected.

▷ Idea of proof:
(of Theorem)



▷ **Thm:** Let A and B be disjoint sets of n points such that $A \cup B$ is not on a line. Then $\mathcal{B}(A,B)$ has a plane subgraph with at least $n+1$ edges.

▷ **Cor:** If $P \subseteq \mathbb{R}^2$ does not lie on a line then $V(P)$ has $K \geq \frac{\delta}{2} + 1$.

▷ If C separates A from B then

$$\delta \leq |A| + |C| - 1 \quad \text{and} \quad |C| \geq |A| + 1.$$

▷ Thm: Let A and B be disjoint sets of n points such that $A \cup B$ is not on a line. Then $B(A,B)$ has a plane subgraph with at least $n+1$ edges.

Outline of Proof: Induction on n. (Base case $n=2$).

Case (i) $\exists$ a line ℓ containing n points.

Use: Lemma: Let A' lie on a line ℓ' and B' have no points on ℓ' with $|A'| \geq |B'|$. Then $B(A',B')$ has a plane subgraph with $|A'| + |B'| - 1$ edges.

Set $\ell' = \ell$, $A' = A \cap \ell \overset{\text{wlog}}{\geq} B \cap \ell$, $B' = B \setminus \ell$.

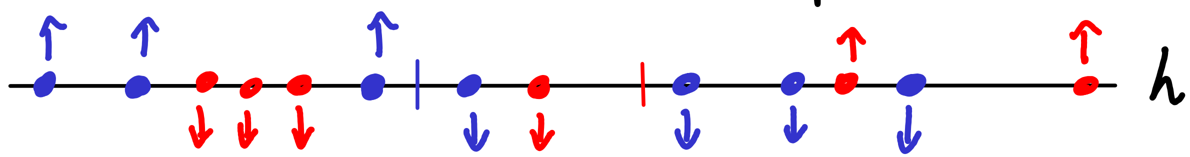
We get $|A'| + |B'| - 1 \geq n-1$ edges.

Need 2 more: ▷ one along ℓ.

▷ $(A \cap \ell = B \cap \ell)$ Can add one far away from ℓ.

▷ Thm: Let A and B be disjoint sets of n points such that $A \cup B$ is not on a line. Then $\mathcal{B}(A,B)$ has a plane subgraph with at least $n+1$ edges.

Case (ii) $\neg\exists$ such a line. Apply the Ham Sandwich Theorem to find a line h with at most half of each set on each side. Assign points on h as follows:



so that each side gets $\lceil \frac{n}{2} \rceil$ of each set.

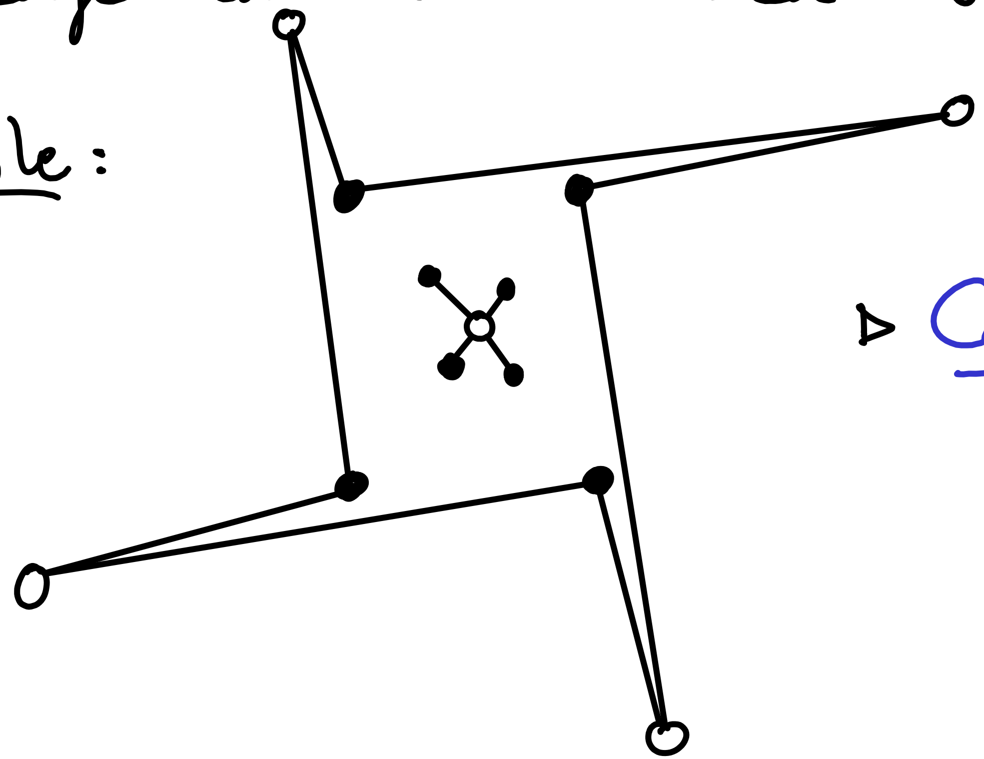
Applying induction on both sides gives $n+2$ edges. We need only delete one to avoid overlaps on h.

▷ **Thm**: For $\ell = 4$, $K \geq \frac{2}{3}\delta$.

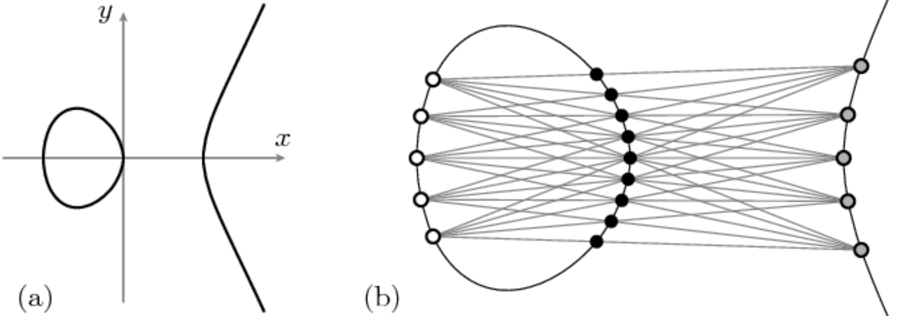
The proof requires the following interesting Lemma:

▷ **Lem**: Let G_1, G_2 be properly coloured plane straight line graph drawings, separated by a line. Then a non-crossing properly coloured edge can be added between them.

Non example:



▷ **Conj**: $K \geq \frac{2}{3}\delta$ always.


y
x
(a)
(b)

Connectedness of Bivisibility Graphs

Lem: $A \cup B$ not on a line. T a triangle with vertices $a \in A$, $b \in B$, $c \in A \cup B$. Then a or b has a neighbour in $B(A,B)$ in $T \setminus \{a, b\}$.

Thm: $B(A,B)$ not on a line. Then $B(A,B)$ has at most one component that is not an isolated vertex.

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Pf:
- Suppose $\exists$ two components with an edge.
- Choose ab, $a'b'$ so that $\mathrm{conv}(\{a,b,a',b'\}) =: C$ is minimal.
- If a, b, a', b' are on a line, use closest pt.
- Otherwise wlog a, b are vertices of C.
- Apply Lemma (with $c = a'$ or b').
- If neighbour is not c then C was not minimal.

$\square$