Bott-Samelson Varieties

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1 Bott towers

A Bott tower of height $N$ is a sequence of $\mathbb{C}P^1$-bundles

$$Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0,$$

such that $Y_0$ is a point and $Y_k = \mathbb{P}(\mathbb{C} \oplus L_k)$ for $1 \leq k \leq N$ where $\mathbb{P}(\cdot)$ denotes complex projectivization, $\oplus$ stands for Whitney sum, $\mathbb{C}$ is a trivial complex line bundle and $L_k$ is a complex line bundle over $Y_{k-1}$.

Note that $L_1 = \mathbb{C}$ since $Y_0 = \{pt\}$. Therefore, $Y_1 = \mathbb{C}P^1$.

Let $C = \{c_{i,j}\}_{1 \leq i < j \leq N}$ be a list of integers. Denote by $Y_C$ the quotient

$$\frac{(\mathbb{C}^2 \setminus 0)^N}{(\mathbb{C}^*)^N},$$

where the $i$th factor of $(\mathbb{C}^*)^N$ acts on $(\mathbb{C}^2 \setminus 0)^N$ by

$$(z_1, w_1, \ldots, z_N, w_N) a_i = (z_1, w_1, \ldots, z_{i-1}, w_{i-1}, z_i a_i, w_i a_i, z_{i+1}, w_{i+1} a_i^{c_{i,i+1}}, \ldots, z_N, w_N a_i^{c_{i,N}}).$$

We denote by $[z_1, w_1, \ldots, z_N, w_N]$ the class of $(z_1, w_1, \ldots, z_N, w_N)$ in $Y_C$.

For $2 \leq n \leq N$, let $C_n = \{c_{i,j}\}_{1 \leq i < j \leq n}$. Then $Y_{C_n}$ is a $\mathbb{C}P^1$-bundle over $Y_{C_{n-1}}$. In fact, let us the line bundle $L(C_{n-1}, c_{1,n}, c_{2,n}, \ldots, c_{n-1,n})$ on $Y_{C_{n-1}}$ by

$$L(C_{n-1}, c_{1,n}, c_{2,n}, \ldots, c_{n-1,n}) = (\mathbb{C}^2 \setminus 0)^{n-1} \times (\mathbb{C}^*)^{n-1} \mathbb{C}.$$
where the $i^{th}$-factor acts by

$$(z_1, w_1, \ldots, z_{n-1}, w_{n-1}), v) a_i = ((z_1, w_1, \ldots, z_{n-1}, w_{n-1}) a_i, a_i^{c_i, n} v)$$

with

$$(z_1, w_1, \ldots, z_{n-1}, w_{n-1}) a_i = (z_1, w_1, \ldots, z_{i-1}, w_{i-1}, z_i a_i, w_i a_i, z_{i+1}, w_{i+1} a_i^{c_i, i+1}, \ldots, z_{n-1}, w_{n-1} a_i^{c_i, n-1}).$$

Then $\mathbb{P}(\mathbb{C} \oplus L_n) = Y_{C_n}$ where $L_n = L(C_{n-1}, c_1, c_2, \ldots, c_{n-1}, c_n)$. Define also $\pi_n : Y_{C_n} \rightarrow Y_{C_{n-1}}$ by

$$\pi([z_1, w_1, \ldots, z_n, w_n]) = [z_1, w_1, \ldots, z_{n-1}, w_{n-1}].$$

Then

$$Y_C = Y_{C_N} \xrightarrow{\pi_N} Y_{C_{N-1}} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} Y_{C_1} \xrightarrow{\pi_1} Y_0 = \{\text{pt}\}$$

where $Y_{C_k} = \mathbb{P}(\mathbb{C} \oplus L_k)$, is a Bott tower of height $N$.

Therefore we have a map

$$\mathbb{Z}^{N(N-1)/2} \rightarrow \{ \text{Bott towers of height } N \}$$

$$C \xrightarrow{\pi} Y_C$$

which is bijective with inverse

$$Y \mapsto \{ \text{coefficients of the first Chern classes of } L_k \}.$$

2 Torus action

The torus $D_C = (\mathbb{C}^*)^N$ with Lie algebra $\mathfrak{d}_C \cong \mathbb{C}^N$, acts on the Boot tower $Y$ by

$$(e^{y_1(d)}, \ldots, e^{y_N(d)}) [z_1, w_1, \ldots, z_N, w_N] = [z_1, e^{-y_1(d)} w_1, \ldots, z_N, e^{-y_N(d)} w_N],$$

where $y_i \in \mathfrak{d}_C^*$ is defined by $y_i((d_1, \ldots, d_N)) = d_i$.

This action of $D_C$ on $Y$ induces an action of $D = (\mathbb{S}^1)^N$ (with Lie algebra $\mathfrak{d} \cong i\mathbb{R}^N \subseteq \mathfrak{d}_C$) on $Y$. In this induced action, the $y_i$ are in $i\mathfrak{d}^*.$
3 Bott-Samelson varieties

For this section we fix the following data:

$G$ a complex semisimple Lie group, which is assumed to be connected and simply connected.

$B$ a fixed Borel subgroup of $G$.

$H$ a fixed maximal torus of $G$.

$K \subseteq G$ a maximal compact subgroup and $T = K \cap H$ a compact maximal torus of $K$.

$X = G/B \cong K/T$ the flag variety. $\mathfrak{h}$ the Lie algebra of $H$.

$\mathfrak{h}^*$ the space dual to $\mathfrak{h}$.

$t \subseteq \mathfrak{h}$ the Lie algebra of $T$.

$\Delta \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{h}$.

$\Delta_+ \subseteq \Delta$ the set of positive roots.

$\Sigma \subseteq \Delta_+$ the system of simple roots.

Let $W = N_G(H)/H$, where $N_G(H)$ is the normalizer of $H$ in $G$. $W$ is the Weyl group of $G$ and it is isomorphic to a real reflection group, generated by the set $S = \{s_\alpha | \alpha \in \Sigma\}$ of simple reflections, i.e. $s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ is a reflection in the hyperplane orthogonal to $\alpha \in \Sigma \subseteq \mathfrak{h}^*$.

The length function $\ell : W \rightarrow \mathbb{N}$ of $W$ relative to the $S$, is defined by $\ell(w) =$ the least number of factors in the decomposition of $w$ in simple reflections, $w = s_{\alpha_1} \cdots s_{\alpha_l}, \alpha_i \in \Sigma$.

Such a decomposition is called reduced if $l = \ell(w)$. $w_0 \in W$ is the unique element of maximal length $r = \ell(w_0)$.

For a set $\beta_1, \ldots, \beta_N$ of simple roots not necessarily distinct, we define the Bott-Samelson variety by

$$\Gamma = \Gamma(\beta_1, \ldots, \beta_N) = \frac{P_{\beta_1} \times \ldots \times P_{\beta_N}}{B^N}$$

where $P_{\beta_i}$ denotes the parabolic subgroup of $G$ corresponding to the root $\beta_i$, and $B^N$ acts on $P_{\beta_1} \times \ldots \times P_{\beta_N}$ by

$$(g_1, \ldots, g_N)(b_1, \ldots, b_N) = (g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{N-1}^{-1} g_N b_N), \quad (b_1, \ldots, b_N) \in B$$

We denote by $[g_1, \ldots, g_N]$ the class of $(g_1, \ldots, g_N)$ in $\Gamma$. The group $B$ acts on $\Gamma$ by

$$b [g_1, g_2, \ldots, g_N] = [bg_1, g_2, \ldots, g_N], \quad (b, g_1, \ldots, g_N) \in B, \quad (b, g_i) \in P_{\beta_i}$$

By restriction we also obtain actions of $H$ and $T$ on $\Gamma$. 

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We define a map $g : \Gamma \rightarrow X$ by multiplication

$$g([g_1, \ldots, g_N]) = g_1 \cdots g_N [B].$$

This map is $T$-equivariant. If we let $w \in W$ with reduced expression $w = s_{\beta_1} \cdots s_{\beta_N}$, then the image of $g$ is the Schubert variety $X_w = BwB/B$, so we have

$$g : \Gamma(\beta_1, \ldots, \beta_N) \rightarrow X_w.$$ 

When $w \in W$ has maximal length $X_w = X$ and

$$g : \Gamma(\beta_1, \ldots, \beta_N) \rightarrow X.$$

**Theorem 3.1** $\Gamma(\beta_1, \ldots, \beta_N) \cong Y_C$ where $C = \{c_{i,j}\}_{1 \leq i < j \leq N}$ with $c_{i,j} = \beta_j(\beta'_{i})$. And the action of $T$ on $\Gamma$ correspond to the action of a subtorus of $D$ on $Y_C$.

### 4 Equivariant cohomology

Let $Y_C$ be the Bott tower corresponding to $C = \{c_{i,j}\}_{1 \leq i < j \leq N}$ and $\Gamma$ the Bott-Samelson variety corresponding to $\beta_1, \ldots, \beta_N$. Then

$$H^*_D(Y_C) = \mathbb{C}[y_1, \ldots, y_N, x_1, \ldots, x_N] / \langle x_i^2 = y_i x_i - \sum_{i<j} c_{i,j} x_i x_j \rangle,$$

$$H^*_T(\Gamma) = \mathbb{C}[\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_N] / \langle x_i^2 = \beta_i x_i - \sum_{i<j} b_{i,j} x_i x_j \rangle,$$

where $\alpha_1, \ldots, \alpha_n$ denotes the simple roots in $\Sigma$ and $b_{i,j} = \beta_j(\beta'_{i})$.

The flag variety $X = G/B$ has equivariant cohomology

$$H^*_T(X) = \mathbb{C}[\alpha_1, \ldots, \alpha_n] - \text{span}\{\psi^w\}_{w \in W}.$$

Let $\mathcal{E} = (\mathbb{Z}/2\mathbb{Z})^N$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \mathcal{E}$, let $\pi_+ (\varepsilon) = \{i \in \{1, \ldots, N\} \mid \varepsilon_i = 1\}$ and $\pi_- (\varepsilon) = \{i \in \{1, \ldots, N\} \mid \varepsilon_i = 0\}$. The length of $\varepsilon \in \mathcal{E}$ is $\ell(\varepsilon) = \# \pi_+ (\varepsilon)$, the cardinality of $\pi_+ (\varepsilon)$. Let $v : \mathcal{E} \rightarrow W$ be the map defined by $v(\varepsilon) = \prod_{k \in \pi_+(\varepsilon)} s_{\beta_k}$.

The multiplication map $g : \Gamma \rightarrow X$ induces a map $g^*$ in cohomology

$$g^* : H^*_T(X) \rightarrow H^*_T(\Gamma)$$

$$\psi^w \mapsto \sum_{\varepsilon \in \mathcal{E}, \ell(\varepsilon) = \ell(w)} \sigma_{\varepsilon}$$

where $\sigma_{\varepsilon} = \prod_{i \in \pi_+(\varepsilon)} x_i$.  

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Example 4.1 For \( C = \{c_{1,2} = 1, c_{1,3} = 2, c_{2,3} = -1\} \), the Bott tower \( Y_C \) has equivariant cohomology

\[
H^*_{D}(Y_C) = \frac{\mathbb{C}[y_1, y_2, y_3, x_1, x_2, x_3]}{\langle x_1^2 = y_1 x_1, x_2^2 = y_2 x_2 + x_1 x_2, x_3^2 = y_3 x_3 + x_2 x_3 + 2 x_1 x_3 \rangle}.
\]

For type \( A_2 \), \( \Gamma = \Gamma(\alpha_1, \alpha_2, \alpha_1) \) has equivariant cohomology

\[
H^*_{T}(\Gamma) = \frac{\mathbb{C}[\alpha_1, \alpha_2, x_1, x_2, x_3]}{\langle x_1^2 = \alpha_1 x_1, x_2^2 = \alpha_2 x_2 + x_1 x_2, x_3^2 = \alpha_1 x_3 + x_2 x_3 + 2 x_1 x_3 \rangle}.
\]

The flag variety \( X = G/B \) in this case has equivariant cohomology

\[
H^*_{T}(X) = \mathbb{C}[\alpha_1, \alpha_2] - \text{span}\{\psi^1, \psi^{s_1}, \psi^{s_2}, \psi^{s_1s_2}, \psi^{s_2s_1}, \psi^{s_1s_2s_1}\}.
\]

The map \( g^* \) maps

- \( \psi^1 \mapsto 1 \), \( \psi^{s_1} \mapsto x_1 + x_3 \), \( \psi^{s_2} \mapsto x_2 \),
- \( \psi^{s_1s_2} \mapsto x_1 x_2 \), \( \psi^{s_2s_1} \mapsto x_2 x_3 \), \( \psi^{s_1s_2s_1} \mapsto x_1 x_2 x_3 \).