1 Definition

Fix the following data:

$V$ a $\mathbb{C}$-vector space of finite dimension $n$ with basis $v_1, \ldots, v_n$.

$V^*$ the dual space of $V$, with basis $X_1, \ldots, X_n$ dual to $v_1, \ldots, v_n$.

$S = \mathbb{C}(X_1, \ldots, X_n)$ the $\mathbb{C}$-algebra of rational functions on $V$.

$W \subseteq GL(V)$ a finite reflection group generated by the reflections $s_1, \ldots, s_r$, acting on $V^*$ by contragredient:

$$wX(v) = X(w^{-1}v) \text{ for } w \in W, X \in V^* \text{ and } v \in V;$$

and on $S$ by extension:

$$w(gh) = (wg)(wh) \quad \text{and} \quad w \left( \frac{g}{h} \right) = \frac{wg}{wh} \text{ for } w \in W \text{ and } g, h \in S.$$ 

$N_W = \langle f \in S; t_w \text{ for } w \in W \mid t_u t_v = t_{uv}, \quad t_w f = (w f) t_w \rangle$ the Nil-Hecke Algebra of $W$.

For each reflection $s_\alpha \in W$, where $\alpha$ denotes the root of the reflection, the divided difference operator $T_\alpha \in N_W$ is defined by

$$T_\alpha = \frac{1}{\alpha} (1 - t_{s_\alpha}).$$

And for any $w \in W$ with reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_l}$ we define $T_w \in N_W$ as

$$T_w = T_{\alpha_1} \cdots T_{\alpha_l}.$$
2 Actions

The Nil-Hecke Algebra acts naturally on the algebra of rational functions $S$ by:

\[ f \cdot g = fg \text{ (polynomial multiplication), for } f \in N_W \text{ and } g \in S; \]
\[ t_w \cdot g = wg \text{ (contragredient action), for } t_w \in N_W \text{ and } g \in S. \]

Therefore, the divided difference operator $T_{\alpha}$ acts on $S$ by

\[ T_{\alpha}f = \frac{f - s_{\alpha}f}{\alpha} \text{ for } f \in S. \]

$N_W$ also acts on the algebra $Z_W$ of global sections of the structure sheaf over the moment graph of $W$,

\[ Z_W = \{ \psi : W \longrightarrow S \mid \psi(u) - \psi(v) \in \alpha S \text{ if } u = s_{\alpha}v \}, \]

by

\[ f \cdot \psi(w) = (wf)\psi(w), \text{ for } f \in N_W \text{ and } \psi \in Z_W; \]
\[ t_u \cdot \psi(w) = \psi(wu), \text{ for } t_u \in N_W \text{ and } \psi \in Z_W. \]

Therefore, the divided difference operator $T_{\alpha}$ acts on $Z_W$ by

\[ T_{\alpha}\psi(w) = \frac{\psi(w) - \psi(ws_{\alpha})}{w_{\alpha}} \text{ for } \psi \in Z_W. \]

3 BGG bases

Recall the definition of the coinvariant algebra $S_W$,

\[ S_W = \frac{\mathbb{C}[X_1, \ldots, X_n]}{(P \in S^W \mid P(0) = 0)}, \]

where $S^W$ denotes the algebra of $W$-invariant polynomials.

Let $Q = \prod \{ \alpha \mid s_{\alpha} \text{ is a reflection} \}$ and $w_0$ be the longest element in $W$ (when expressed as a reduced word in the generators).

**Theorem 3.1 (BGG).** \( \{ P_w \}_{w \in W} \) is a basis of $S_W$ as a $\mathbb{C}$-vector space, where $P_{w_0} = \frac{1}{|W|}Q$ and $P_w = T_{w^{-1}w_0}P_{w_0}$. 
Theorem 3.2. \( \{\psi^w\}_{w \in W} \) is a basis of \( Z_W \) where

\[
\psi^w_0(w) = \begin{cases} 
Q, & \text{if } w = w_0; \\
0, & \text{if } w \neq w_0.
\end{cases}
\]

and \( \psi^w = T_{w^{-1}}w\psi^w_0 \).

Theorem 3.3. The map

\[
\phi : S_W \rightarrow Z_W
\]

\[
P_w \mapsto \psi^w
\]

is an algebra isomorphism.

4 Example

Let \( V = \mathbb{C} - \text{span}\{v_1, v_2, v_3\} \) and \( V^\ast = \mathbb{C} - \text{span}\{X_1, X_2, X_3\} \).

Let \( s_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( s_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \).

and \( W = \langle s_1, s_2 \rangle = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \subseteq GL(V) \). Since \( w_0 = s_1s_2s_1 \) is also a reflection, we will also denote it as \( s_3 \),

\[
s_3 = s_1s_2s_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

The action of \( W \) on \( V^\ast \) is given by the inverse transpose of the above matrices. But since in this case \( s_i^{-t} = s_i \) for \( i = 1, 2, 3 \); the action is the same as in \( V \).

Let us denote by \( \alpha_i \) the root in \( V^\ast \) corresponding to \( s_i \) for \( i = 1, 2, 3 \). Then

\[
\alpha_1 = X_1 - X_2, \quad \alpha_2 = X_2 - X_3, \quad \alpha_3 = X_1 - X_3.
\]

The coinvariant algebra is

\[
S_W = \frac{\mathbb{C}[X_1, X_2, X_3]}{\langle X_1 + X_2 + X_3, X_1X_2 + X_1X_3 + X_2X_3, X_1X_2X_3 \rangle}.
\]

As a vector space,
\[ S_W = \mathbb{C} - \text{span} \left\{ 1, X_1, X_2, X_1^2, X_1X_2, X_2^2X_2 \right\} \]

The BGG basis theorem gives us

\[
P_{w_0} = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 = X_1^2 X_2,
\]

\[
P_{s_1s_2} = \frac{1}{3} \alpha_2 \alpha_3 = X_1 X_2,
\]

\[
P_{s_2s_1} = \frac{1}{3} \alpha_1 \alpha_3 = X_1^2,
\]

\[
P_{s_1} = \frac{1}{3} (\alpha_1 + \alpha_3) = X_1,
\]

\[
P_{s_2} = \frac{1}{3} (\alpha_2 + \alpha_3) = X_1 + X_2,
\]

\[
P_c = 1,
\]

as a basis for \( S_W \), i.e.

\[
S_W = \mathbb{C} - \text{span} \left\{ 1, X_1, X_1 + X_2, X_1^2, X_1X_2, X_2^2X_2 \right\}
\]

which clearly agrees with the basis above.

For \( Z_W \) we get the following basis elements

\[
\psi^{s_{00}} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_1 \alpha_2 \alpha_3 \end{pmatrix},
\]

\[
\psi^{s_{1s_2}} = \begin{pmatrix} 0 & 0 \\ -\alpha_1 \alpha_3 & 0 \end{pmatrix},
\]

\[
\psi^{s_{2s_1}} = \begin{pmatrix} 0 & 0 \\ -\alpha_2 \alpha_3 & 0 \end{pmatrix},
\]

\[
\psi^{s_1} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_3 & 0 \end{pmatrix},
\]

\[
\psi^{s_2} = \begin{pmatrix} \alpha_2 & 0 \\ \alpha_3 & 0 \end{pmatrix},
\]

\[
\psi^c = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.
\]