1 Pullbacks

The pullback of two morphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ consists of an object $X \times_A Y$ and two morphisms $X \times_A Y \rightarrow X$ and $X \times_A Y \rightarrow Y$, satisfying the following universal property:

$$
\begin{array}{ccc}
  Z & \rightarrow & X \\
  \downarrow & & \downarrow f \\
 X \times_A Y & \rightarrow & Y \\
  \downarrow & & \downarrow g \\
  Y & \rightarrow & A
\end{array}
$$

For small categories we can explicitly write

$$X \times_A Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

2 Fibrations

A map $p : E \rightarrow B$ is a (Hurewicz) fibration if it satisfies the homotopy lifting property (HLP) with respect to all spaces. This means that if $Y$ is a space and $h \circ i_0 = p \circ f$ in the diagram

$$
\begin{array}{ccc}
  Y & \overset{f}{\rightarrow} & E \\
  i_0 \downarrow & & \downarrow p \\
  Y \times I & \overset{h}{\rightarrow} & B
\end{array}
$$
then there exists $\tilde{h}$ that makes the diagram commute.

Here $I = [0, 1]$ and $i_0 : y \mapsto (y, 0)$.

$E$ is the total space, $B$ is the base space and $F_b = p^{-1}(b)$ for $b \in B$ is the fiber over $b$ of the fibration $p$.

All fibers in a path component of $B$ are homotopy equivalent. Thus, when $B$ is path connected, all fibers are homotopy equivalent and we regard them as the fiber (unique up to homotopy) of the fibration $p$, denoted simply as $F$.

3 Replacing a map by a fibration

An arbitrary map $f : X \to Y$ can be decomposed into a homotopy equivalence followed by a fibration:

$$
\begin{align*}
X \xrightarrow{\nu} N_f \xrightarrow{\rho} Y \\
x \mapsto (x, C_{f(x)}) \\
(x, \beta) \mapsto \beta(0),
\end{align*}
$$

$\nu$ is an homotopy equivalence, $\rho$ a fibration, and $f = \rho \circ \nu$. Where

$N_f = X \times_f Y^I = \{(x, \beta) \in X \times Y^I : f(x) = \beta(1)\}$, the pullback of $\xymatrix{Y^I \ar[r]^{p_1} & Y \ar[l]^{f} & X}$,

is the mapping path space of $f$, with

$Y^I = Hom(I, Y)$ and $p_1 : \beta \mapsto \beta(1)$,

and

$C_{f(x)} : t \mapsto f(x)$ (the constant map).

4 Homotopy Fiber

The homotopy fiber $F_f$ of a map $f : X \to Y$, is the fiber of the associated fibration $\rho : N_f \to Y$.

Fix a basepoint $* \in Y$, take 0 to be the basepoint of $I$ and let $PY$ be the subset of $Y^I$ consisting in based maps. Then
\[ F_f = \rho^{-1}(\ast) \]
\[ = \{(x, \beta) \in N_f : \beta \in PY\} \]
\[ = \{(x, \beta) \in X \times Y^I : f(x) = \beta(1) \land \beta(0) = \ast\} \]
\[ = X \times_f PY, \] the pullback of \[ PY \xrightarrow{p_1} Y \xleftarrow{f} X. \]

In this case, the universal property of the pullback is equivalent to a bijective correspondence between lifts and null-homotopies, i.e.

**Proposition 4.1** Given spaces \( X, Y, Z \) and maps \( f : X \to Y, \) \( g : Z \to X, \) there is a (lifted) map \( \tilde{g} : Z \to F_f \) such that \( \pi_1 \circ \tilde{g} = g, \) where \( \pi_1 : F_f \to X \) is the canonical projection, iff \( f \circ g \simeq C_\ast \) (null-homotopic).

![Diagram](attachment:image.png)

**Proof**

(\( \Rightarrow \)) Let \( h = \pi_2 \circ \tilde{g} : Z \to PY. \) If we consider \( h \) as a map of \( Z \times I \) into \( Y, \) then it must satisfy \( h(z, 0) = C_\ast(z) = \ast \) and \( h(z, 1) = (f \circ g)(z), \) because of the commutativity of the diagram and the definitions of the spaces involved. Thus, \( h : f \circ g \simeq C_\ast. \)

(\( \Leftarrow \)) Let \( h : f \circ g \simeq C_\ast. \) Then \( \tilde{g} = g \times h \) is the desired lift.

\[ \square \]

**5 Fibration sequences**

The **fibration sequence** of a fibration \( p : E \to B \) with fiber \( F, \) is the sequence

\[ F \hookrightarrow E \xrightarrow{p} B. \]

**Examples**

- For a map \( f : X \to Y, \) we associate the fibration sequence \( F_f \hookrightarrow N_f \xrightarrow{\rho} Y. \)
• For a group $G$ with a subgroup $H \trianglelefteq G$, the sequence $G/H \hookrightarrow BH \xrightarrow{\rho} BG$ is a fibration sequence, where $BH$ and $BG$ are the classifying spaces of $H$ and $G$ respectively.

6 References

