1 Reflections

Let \( V \) be a complex vector space of dimension \( n \), and let \( s : V \rightarrow V \) be a linear transformation fixing pointwise a subspace of codimension 1, i.e.

\[
\dim V^s = n - 1, \quad \text{where } V^s = \{ v \in V | s(v) = v \}.
\]

**Proposition 1.1** The matrix of \( s \) is similar to

\[
\begin{bmatrix}
1 & 1 & & & \\
& 1 & 1 & & \\
& & \ddots & \ddots & \\
& & & 1 & 1 \\
& & & * & 1
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
1 & & & & * \\
& 1 & & & 1 \\
& & \ddots & \ddots & \\
& & & 1 & 1 \\
& & & & 1
\end{bmatrix}
\]

with \( * \in \mathbb{C} \).

**Proof**

Let \( \mathcal{B} = \{ v_1, \ldots, v_{n-1} \} \) be a basis of \( V^s \). The matrix of \( s|_{V^s} \) in \( \mathcal{B} \) is the identity matrix, i.e. \( [s]_{\mathcal{B}^s} = I_{n-1} \). Extend \( \mathcal{B} \) to a basis \( \mathcal{B}' \) of \( V \) by adding an extra vector \( v_n \in V \setminus V^s \) to \( \mathcal{B} \).

Suppose \( s(v_n) = b_1 v_1 + \cdots + b_n v_n \neq v_n \). Then

\[
[s]_{\mathcal{B}'} = \begin{bmatrix}
1 & b_1 \\
1 & b_2 \\
& \ddots & \vdots \\
& & 1 & b_{n-1} \\
& & & 1 & b_n
\end{bmatrix}
\]
Consider \( v = c_1 v_1 + \cdots + c_{n-1} v_{n-1} + v_n \). Then
\[
  s(v) = c_1 v_1 + \cdots + c_{n-1} v_{n-1} + b_1 v_1 + \cdots + b_n v_n
  = (c_1 + b_1) v_1 + \cdots + (c_{n-1} + b_{n-1}) v_{n-1} + b_n v_n
\]

If \( b_n \neq 1 \), we take \( c_i = b_i / (b_n - 1) \) so that \( s(v) = b_n v \).

Define a new basis \( B'' = \{v_1, \ldots, v_{n-1}, v\} \). With respect to this basis the matrix of \( s \),
\[
  [s]_{B''} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
 & 1 & & b_n \\
 & & \ddots & \vdots \\
 & & & 1 & b_{n-1} \\
& & & & 1 \\
\end{bmatrix},
\]
has the desired form.

If \( b_n = 1 \), then the matrix
\[
  [s]_{B'} = \begin{bmatrix}
1 & b_1 \\
1 & b_2 \\
& \ddots & \vdots \\
& & 1 & b_{n-1} \\
& & & 1 \\
\end{bmatrix}
\]
is not diagonalizable since its only eigenvalue is 1 and the corresponding eigenspace has
dimension \( n - 1 \) (less than the size of the matrix). By Jordan normal form, this means that
the number of Jordan blocks corresponding to the eigenvalue 1 is \( n - 1 \) and the matrix is
similar to
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
& \ddots & \ddots & \ddots \\
& & 1 & 1 \\
& & & 1 \\
\end{bmatrix}
\]

\( \square \)

Remark 1.2 If in the above proof \( b_n = 0 \) then
\[
  [s]_{B''} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
& 1 & \cdots & 1 \\
& & \ddots & \ddots \\
& & & 1 & 0 \\
\end{bmatrix}
\]
and $s$ is a projection. If we impose $s$ to be invertible, i.e. $s \in GL(V)$, then $b_n$ can’t be 0. For if so, $s(v_n) \neq v_n$ but $s(s(v_n)) = s(v_n)$, and $s$ will not be injective (projections are not injective).

Furthermore, if we impose $s$ to be of finite order $k > 1$, then

$$b_1(1 + b_n + \cdots + b_n^k)v_1 + \cdots + b_{n-1}(1 + b_n + \cdots + b_n^{k-1})v_{n-1} + b_n^{k+1}v_n = s^k(v_n)$$

$$= v_n = b_1v_1 + \cdots + b_nv_n,$$

implies $b_n$ is a $k^{th}$ root of unity.

**Definition 1.3** Let $V$ be a complex vector space. A (complex) reflection in $V$ is a linear transformation $s \in GL(V)$ of finite order.

**Corollary 1.4** The matrix of a reflection is similar to

$$\begin{bmatrix}
1 & 1 \\
& \ddots \\
& & 1 \\
& & \ast
\end{bmatrix}$$

where $1 \neq \ast \in \mathbb{C}$ is a root of unity.

Let $V$ be a complex vector space. A **complex reflection group** is a subgroup $W \subseteq GL(V)$ generated by reflections.

Let $V^*$ be the dual space of $V$. The group $W$ acts on $V^*$ by

$$wX(v) = X(w^{-1}v), \quad \text{for } w \in W, X \in V^* \text{ and } v \in V,$$

the **contragredient** action.

If $A$ is the matrix of the action of a reflection $s \in W$ on $V$ with respect to a fixed basis of $V$, then the matrix of the action of the reflection on $V^*$ with respect to the dual basis is $A^{-t}$, the inverse transpose of $A$.

Recall $(A^{-1})^t = (A^t)^{-1}$ and therefore there is no ambiguity in writing the symbol $A^{-t}$ when referring to the inverse transpose matrix.
Lemma 1.5 Let $V$ be a vector space with bases $B = \{v_1, \ldots, v_n\}$, $B' = \{v'_1, \ldots, v'_n\}$, and change of basis matrix $P : B \rightarrow B'$. Let $V^*$ be the dual space of $V$ with bases $C = \{X_1, \ldots, X_n\}$ dual to $B$, $C' = \{X'_1, \ldots, X'_n\}$ dual to $B'$ and change of basis matrix $Q : C \rightarrow C'$.

Diagrammatically

\[
\begin{array}{c c}
B = \{v_1, \ldots, v_n\} & \text{is the dual basis of} & C = \{X_1, \ldots, X_n\} \\
\downarrow P & & \downarrow Q \\
B' = \{v'_1, \ldots, v'_n\} & \text{is the dual basis of} & C' = \{X'_1, \ldots, X'_n\}
\end{array}
\]

Then $Q = P^{-t}$.

Proof

Write $v_i \in B$ in basis $B'$, $v_i = p_{1i}v'_1 + p_{2i}v'_2 + \cdots + p_{ni}v'_n$. Then

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}
\]

is the change of basis matrix from $B$ to $B'$, i.e. $P[v]_B = [v]_{B'}$.

Note that $X'_j(v_i) = X'_j(p_{1i}v'_1 + \cdots + p_{ni}v'_n) = p_{ji}$, for $X'_j \in C'$. Then for $v \in V$ written in basis $B$, $v = b_1v_1 + \cdots + b_nv_n$ we have

\[
X'_j(v) = X'_j(b_1v_1 + \cdots + b_nv_n) = b_1p_{j1} + \cdots + b_np_{jn} = p_{j1}X_1(v) + \cdots + p_{jn}X_n(v).
\]

So $X'_j = p_{j1}X_1 + \cdots + p_{jn}X_n$. And therefore

\[
Q^{-1} = \begin{bmatrix}
p_{11} & p_{21} & \cdots & p_{n1} \\
p_{12} & p_{22} & \cdots & p_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1n} & p_{2n} & \cdots & p_{nn}
\end{bmatrix} = P^t.
\]

Thus, $Q = P^{-t}$.

\[\square\]

Let $s \in W \subseteq GL(V)$ be a reflection. A root of $s$ is an eigenvector $\alpha \in V$ associated to the nontrivial eigenvalue of $s$. 

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Proposition 1.6 Let $s \in W \subseteq GL(V)$ be a reflection in $V$. Then $s$ is also a reflection in $V^*$, and if $\alpha^\vee$ is a root of $s$ in $V^*$ then $V^s = \ker(\alpha^\vee)$.

Proof

Let $A$ be the matrix of $s$ in some fixed basis of $V$. By corollary 1.4, $A$ is similar to a matrix of the form

$$D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & \cdot \\ & & & 1 \\ & & & & c \end{bmatrix}$$

with $c \in \mathbb{C}$, i.e. there exist an invertible matrix $P$ such that $PAP^{-1} = D$.

Then $P^{-t}A^tP^t = (PAP^{-1})^t = D^t = D$, so

$$P^{-t}A^{-t}P^t = (P^{-t}A^tP^t)^{-1} = D^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & \cdot \\ & & & 1 \\ & & & & c^{-1} \end{bmatrix}$$

Thus, $A^{-t}$ is a reflection in $V^*$.

Let $\alpha^\vee \in V^*$ be a root of $s$ in $V^*$, i.e. an eigenvector of $A^{-t}$ associated to the nontrivial eigenvalue $c^{-1}$. A vector $v = (b_1, \ldots, b_n) \in V$ is in $V^s$ iff $v = Av$, i.e.

$$\begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & \cdot \\ & & & 1 \\ & & & & c \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ cb_n \end{bmatrix},$$

which happens iff $cb_n = b_n$ iff $0 = b_n = \alpha(v)$ iff $v \in \ker(\alpha)$.

Here $\alpha^\vee = (0, 0, \cdots, 1)$ in the basis where $D^{-1}$ is diagonal, which is dual to the basis where $D$ is diagonal by lemma 1.5.

$\square$

Let $V$ be a complex vector space. Let $[\phantom{1}, \phantom{1}]$ denote the dot product in $V$ and $\langle \phantom{1}, \phantom{1} \rangle$ denote the pairing $V \times V^* \rightarrow \mathbb{C}$. 

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Proposition 1.7  A linear transformation \( s \in GL(V) \) is a complex reflection of order \( r \) and root \( \alpha \) iff

\[
s(v) = v - \langle v, \alpha^\vee \rangle \alpha \quad \text{where} \quad \alpha^\vee = \frac{(1 - \mu)[\alpha,\alpha]}{[\alpha,\alpha]} \in V^*\]

and \( \mu \in \mathbb{C} \) is a primitive \( r^{th} \)-root of unity.

**Proof**

Assume \( s \in GL(V) \) is a complex reflection of order \( r \) and root \( \alpha \). Let \( v = (v_1,\ldots,v_n) \in V \) in the basis where the matrix of \( s \) is diagonal and take \( \alpha = (0,\ldots,0,1) \). Then

\[
s(v_1,\ldots,v_n) = (v_1,\ldots,\mu v_n) = (v_1,\ldots,v_n) - (1 - \mu)v_n(0,\ldots,0,1) = v - \langle v, \alpha^\vee \rangle \alpha.
\]

Now assume \( s \in GL(V) \) such that

\[
s(v) = v - \langle v, \alpha^\vee \rangle \alpha \quad \text{where} \quad \alpha^\vee = \frac{(1 - \mu)[\alpha,\alpha]}{[\alpha,\alpha]} \in V^*
\]

and \( \mu \in \mathbb{C} \) is a primitive \( r^{th} \)-root of unity. Then

\[
v \in V^s \iff v = s(v) = v - \langle v, \alpha^\vee \rangle \alpha \iff \langle v, \alpha^\vee \rangle = 0 \iff [v,\alpha] = 0 \iff v \in \alpha^\perp
\]

where \( \alpha^\perp = \{v \in V | [v,\alpha] = 0\} \) is the orthogonal complement of the line spanned by \( \alpha \). This is a codimension one subspace. Thus, \( s \) is a reflection.

This reflection has root \( \alpha \) since \( \alpha - \langle \alpha, \alpha^\vee \rangle \alpha = \mu \alpha \). Moreover,

\[
s^k(v) = v - (1 + \mu + \mu^2 + \cdots + \mu^{k-1})\langle v, \alpha^\vee \rangle \alpha
\]

implies that the order of \( s \) is \( r \).

\[\square\]

Proposition 1.7 gives us an equivalent definition of a reflection. Using this definition, we get an alternative proof of Proposition 1.6 as follows

**Proof** of Proposition 1.6 (alternative)

Let \( s \in W \) be a reflection in \( V \), let \( v \in V \) and \( X \in V^* \). Then \( s(v) = v - \langle v, \alpha^\vee \rangle \alpha \) and

\[
s^{-1}X(v) = X(sv) = X(v - \langle v, \alpha^\vee \rangle \alpha) = X(v) - \langle v, \alpha^\vee \rangle X(\alpha),
\]
or, with the pairing notation

\[ s^{-1}X(v) = (sv, X) = \langle v - \langle v, \alpha^\vee \rangle \alpha, X \rangle = \langle v, X \rangle - \langle v, \alpha^\vee \rangle \langle \alpha, X \rangle. \]

So \( s^{-1}X = X - \langle X, \alpha \rangle \alpha^\vee \). This is, \( s^{-1}X \), and therefore \( s \), are reflections in \( V^* \).

Let us show that \( V^s \subseteq \ker(\alpha^\vee) \). Let \( v \in V^s \), i.e. \( s(v) = v \), and let \( X \in V^* \). Then

\[ X(sv) = s^{-1}X(v) = X(v) - \langle v, \alpha^\vee \rangle X(\alpha). \]

But \( sv = v \) since \( v \in V^s \). Then

\[ X(v) = X(v) - \langle v, \alpha^\vee \rangle X(\alpha), \]

which implies that \( \langle v, \alpha^\vee \rangle X(\alpha) = 0 \). And since \( X \) is arbitrary, it follows that \( \langle v, \alpha^\vee \rangle = 0 \), i.e. \( v \in \ker(\alpha^\vee) \).

For the other containment, let \( v \in \ker(\alpha^\vee) \), i.e. \( \langle v, \alpha^\vee \rangle = 0 \), and let \( X \in V^* \). Then

\[ X(sv) = X(v) - \langle v, \alpha^\vee \rangle X(\alpha) = X(v). \]

But since \( X \) is arbitrary, this means that \( sv = v \), i.e. \( v \in V^s \).

\[ \square \]

2 Divided Difference Operators

For this section we fix the following data:

\( G \) a complex semisimple Lie group, which is assumed to be connected and simply connected.
\( B \) a fixed Borel subgroup of \( G \).
\( H \) a fixed maximal torus of \( G \).
\( \mathfrak{h} \) the Lie algebra of \( H \).
\( \mathfrak{h}^* \) the space dual to \( \mathfrak{h} \).
\( \Delta \subseteq \mathfrak{h}^* \) the root system of \( \mathfrak{h} \).
\( \Delta_+ \subseteq \Delta \) the set of positive roots.
\( \Sigma \subseteq \Delta_+ \) the system of simple roots.

Let \( W = N_G(H)/H \), where \( N_G(H) \) is the normalizer of \( H \) in \( G \). \( W \) is the Weyl group of \( G \) and it is isomorphic to a real reflection group, generated by the set \( S = \{ s_\alpha | \alpha \in \Sigma \} \) of simple reflections, i.e. \( s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \) is a reflection in the hyperplane orthogonal to
\[ \alpha \in \Sigma \subseteq \mathfrak{h}^*. \]

The **length** function \( \ell : W \rightarrow \mathbb{N} \) of \( W \) relative to the \( S \), is defined by \( \ell(w) = \) the least number of factors in the decomposition of \( w \) in simple reflections, \( w = s_{\alpha_1} \cdots s_{\alpha_l}, \alpha_i \in \Sigma. \)

Such a decomposition is called reduced if \( l = \ell(w). \ w_0 \in W \) is the unique element of maximal length \( r = \ell(w_0). \)

Let \( S(\mathfrak{h}^*) \) be the symmetric algebra of \( \mathfrak{h}^* \). \( S(\mathfrak{h}^*) \) is isomorphic to the algebra of polynomial functions on \( \mathfrak{h}. \)

For each root \( \alpha \in \Delta \) the **divided difference operator** \( \Delta_{\alpha} : S(\mathfrak{h}^*) \rightarrow S(\mathfrak{h}^*) \) is defined by

\[
\Delta_{\alpha}(f) = \frac{f - s_\alpha(f)}{\alpha} \quad \text{for} \quad f \in S(\mathfrak{h}^*). 
\]

The following proposition allow us to well define a divided difference operator \( \Delta_w \) for any \( w \in W. \)

**Proposition 2.1** Let \( w = s_{\alpha_1} \cdots s_{\alpha_l}, \alpha_i \in \Sigma. \)
1. If \( \ell(w) < l \), then \( \Delta_{\alpha_1} \cdots \Delta_{\alpha_l} = 0. \)
2. If \( \ell(w) = l \), then \( \Delta_{\alpha_1} \cdots \Delta_{\alpha_l} \) depends only on \( w \) and not on its expression \( w = s_{\alpha_1} \cdots s_{\alpha_l}. \)

We define then \( \Delta_w = \Delta_{\alpha_1} \cdots \Delta_{\alpha_l} \) where \( s_{\alpha_1} \cdots s_{\alpha_l} \) is a reduced expression of \( w. \)

The divided difference operators induce well defined operators \( \Delta_w \) in the **coinvariant algebra** \( S_W = S(\mathfrak{h}^*)/I_W \) of \( W. \)

The following theorem specifies a basis \( \{P_w|w \in W\} \) of \( H^*(G/B) \) dual to the basis \( \{Z_w|w \in W\} \) of \( H_*(G/B) \) consisting of Schubert classes.

**Theorem 2.2** Berstein, Gel’fand, Gel’fand (1973)

1. Let \( w_0 \in W \) be the element of maximal length \( r = \ell(w_0). \) Then

\[
P_{w_0} = \rho^r/r! = |W|^{-1} \prod_{\alpha \in \Delta_+} \alpha \quad \text{where} \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.
\]

2. If \( w \in W, \) then \( P_w = \Delta_{w_0^{-1}w_0}P_{w_0}. \)
In other words, the previous theorem gives an isomorphism

\[ H^*(G/B) \xrightarrow{\sim} S_W \]

\[ Z_w \rightarrow P_w \]