Let $V$ be a $\mathbb{C}$-vector space of finite dimension $n$ with basis $v_1, \ldots, v_n$. Let $V^* = \text{Hom}(V, \mathbb{C})$ be the dual space of $V$, with basis $X_1, \ldots, X_n$ dual to $v_1, \ldots, v_n$. The symmetric algebra $S = S(V^*)$ of $V^*$ is the $\mathbb{C}$-algebra $\mathbb{C}[X_1, \ldots, X_n]$ of polynomial functions on $V$.

Note that this is distinct from the algebra $\mathbb{C}[v_1, \ldots, v_n]$ of polynomials in $V$ (or in the symbols $v_1, \ldots, v_n$). This algebra correspond to $S(V)$, the symmetric algebra of $V$.

The algebra $S$ has a natural grading by degree,

$$S = \bigoplus_{j \geq 0} S_j,$$

where $S_r$ denotes the subspace of homogeneous polynomials of degree $r$.

The general linear group $GL(V)$ acts on $V^*$ by:

$$gX(v) = X(g^{-1}v) \quad \text{for} \quad g \in GL(V), \ X \in V^* \text{ and } v \in V,$$

the contragredient action. This action extends to $S$ by:

$$g(PQ) = (gP)(gQ) \quad \text{for} \quad g \in GL(V) \text{ and } P, Q \in S.$$

Note that this action is linear and preserves the degree in $S$, i.e. $gS_j = S_j$.

Let $W$ be a finite subgroup of $GL(V)$ with the inherited action on $S$. The invariant algebra of $W$ is $S^W = \{P \in S \mid \text{if } w \in W \text{ then } wP = P\}$. This is a $\mathbb{C}$-subalgebra of $S$. 

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Since the $W$-action on $S$ preserves degree, we have a decomposition

$$S^W = \bigoplus_{j \geq 0} S^W_j$$

into homogeneous invariants. Note that if the $W$-action didn’t preserve the degree on $S$, then some inhomogeneous invariant polynomials could not be in $\bigoplus_{j \geq 0} S^W_j$.

Let $S^W_+$ be the (maximal) ideal of $S^W$ consisting of polynomials with zero constant term, i.e.

$$S^W_+ = \{ P \in S^W \mid P(0) = 0 \} = \bigoplus_{j > 0} S^W_j,$$

and $F$ the ideal of $S$ generated by $S^W_+$,

$$F = \langle S^W_+ \rangle = \{ P_1 J_1 + \cdots + P_k J_k \mid k \in \mathbb{N}, P_i \in S, J_i \in S^W_+ \}.$$

Any ideal of $S = \mathbb{C}[X_1, \ldots, X_n]$ is, by Hilbert’s basis theorem [Hil], finitely generated. So we can write $F = \langle I_1, \ldots, I_r \rangle$ and assume $I_1, \ldots, I_r$ are $W$-invariants and homogeneous since $F = \langle S^W_+ \rangle$ and the $W$-action preserves degree. Therefore we have the decomposition

$$F = \bigoplus_{j > 0} F \cap S_j = \bigoplus_{j > 0} \mathbb{C} - \text{span}\{ P I_i \mid 1 \leq i \leq r, \deg(P I_i) = j \text{ and } P \text{ is monic monomial in } S \}$$

into homogeneous pieces.

The **coinvariant algebra** $S_W$ of $W$ is the quotient $\mathbb{C}$-algebra $S/F$. A maximal set of monic monomials $P_1, \ldots, P_s$ in $S$ that cannot be expressed as linear combinations of the others using the relations $I_i = 0$ for $1 \leq i \leq r$, form a $\mathbb{C}$-basis (modulo $F$) of $S_W$.

Next we will compute the $\mathbb{C}$-dimension of $S^W$ with the aid of another vector space. Let $\mathbb{C}(V) = \mathbb{C}(X_1, \ldots, X_n)$ denote the field of fractions of $S$, i.e. the field of rational functions $P/Q$ with $P, Q \in S$ and $Q \neq 0$.

The action of $W$ on $S$ extends to an action on $\mathbb{C}(V)$ by $w(P/Q) = wP/wQ$.

**Lemma 0.1** The $W$-fixed field $\mathbb{C}(V)^W$ is the field of fractions

$$\mathbb{C}(S^W) = \mathbb{C}(I_1, \ldots, I_r)$$

of $S^W$. In particular $\mathbb{C}(V)$ is a vector space over $\mathbb{C}(S^W)$ of dimension $|W|$.
Proof

Let \( P/Q \in \mathbb{C}(V)^W \), i.e. if \( w \in W \) then \( w(P/Q) = P/Q \).

Multiplying numerator and denominator by \( \prod_{w \neq 1} wQ \), if necessary, we can assume that if \( w \in W \) then \( wQ = Q \) and hence \( wP = P \). This means \( P/Q \in \mathbb{C}(S^W) \).

Artin’s Theorem [Ing] says that \( \mathbb{C}(V)/\mathbb{C}(S^W) \) is a Galois extension with Galois group \( W \) and degree \( [\mathbb{C}(V) : \mathbb{C}(S^W)] = |W| \), which means that \( \mathbb{C}(V) \) is a vector space over \( \mathbb{C}(S^W) \) of dimension \( |W| \).

□

Lemma 0.2 If \( W \) is generated by reflections and \( P_1, \ldots, P_s \) are homogeneous, linearly independent polynomials in the vector space \( S \) over \( \mathbb{C} \), then \( P_1, \ldots, P_s \) are linearly independent in the vector space \( \mathbb{C}(V) \) over \( \mathbb{C}(S^W) \).

For a proof of this lemma, see [Che, Lemma 3].

Lemma 0.3 Let \( P_1, \ldots, P_s \) be a \( \mathbb{C} \)-basis of \( S^W \) consisting of (residues classes modulo \( F \) of) homogeneous polynomials in \( S \). Then every polynomial \( P \in S \) has a unique expression of the form \( P = U_1P_1 + U_2P_2 + \cdots + U_sP_s \), where \( U_i \in S^W \) for \( 1 \leq i \leq s \).

Proof

It is enough to prove the statement for \( P \in S \) homogeneous. The proof is by induction on \( \deg(P) \).

If \( \deg(P) = 0 \), \( P \in \mathbb{C} \) and \( P = P1 \) is of the give form (note that \( 1 \in \{ P_1, \ldots, P_s \} \) since otherwise \( 1 \in F \), i.e. \( F = S \) and \( S/F = \{ 0 \} \)).

Now assume \( \deg(P) > 0 \). Since \( P_1, \ldots, P_s \) is a basis of \( S/F \) and \( P \) is homogeneous, there exist \( a_1, \ldots, a_s \in \mathbb{C} \) and \( Q \in F \) homogeneous such that

\[
P = a_1P_1 + a_2P_2 + \cdots + a_sP_s + Q.
\]

But \( F \) is generated by \( I_1, \ldots, I_r \) as an ideal of \( S \), so there are \( Q_1, \ldots, Q_r \in S \) homogeneous such that \( Q = Q_1I_1 + Q_2I_2 + \cdots + Q_rI_r \) and \( \deg(Q_j) < \deg(P) \) for \( 1 \leq j \leq r \).

Then, by induction, each \( Q_j \) can be expressed in the desired form as \( Q_j = U_{j1}P_1 + U_{j2}P_2 + \cdots + U_{js}P_s \) for some \( U_{j1}, \ldots, U_{js} \in S^W \) and hence
\[
P = a_1P_1 + a_2P_2 + \cdots + a_sP_s + Q_1I_1 + Q_2I_2 + \cdots + Q_rI_r
= a_1P_1 + a_2P_2 + \cdots + a_sP_s + (U_{11}P_1 + U_{12}P_2 + \cdots + U_{1s}P_s)I_1
+ (U_{21}P_1 + U_{22}P_2 + \cdots + U_{2s}P_s)I_2 + \cdots + (U_{r1}P_1 + U_{r2}P_2 + \cdots + U_{rs}P_s)I_r
= (a_1 + U_{11}I_1 + U_{21}I_2 + \cdots + U_{r1}I_r)P_1 + (a_2 + U_{12}I_1 + U_{22}I_2 + \cdots + U_{r2}I_r)P_2
+ \cdots + (a_s + U_{1s}I_1 + U_{2s}I_2 + \cdots + U_{rs}I_r)P_s,
\]
where \(a_i + U_{1i}I_1 + \cdots + U_{ri}I_r \in S^W\) for \(1 \leq i \leq s\).

The uniqueness follows from lemma 0.2.

\[\square\]

**Proposition 0.4** \(\dim_{\mathbb{C}}(S_W) = |W|\).

**Proof**

From lemma 0.1 we know that \(|W| = \dim_{\mathbb{C}(S^W)}\mathbb{C}(V)\).

The basis \(P_1, \ldots, P_s\) of \(S_W\) consisting of (residues classes modulo \(F\) of) homogeneous polynomials in \(S\) is linearly independent in \(\mathbb{C}(V)\) over \(\mathbb{C}(S^W)\) by lemma 0.2.

We show that \(P_1, \ldots, P_s\) also span \(\mathbb{C}(V)\) over \(\mathbb{C}(S^W)\). Let \(P/Q \in \mathbb{C}(V)\), multiplying numerator and denominator by \(\prod_{w \neq 1} wQ\), if necessary, we can assume that \(Q \in S^W\). By lemma 0.3, \(P = U_1P_1 + U_2P_2 + \cdots + U_sP_s\) for some \(U_1, U_2, \ldots, U_s \in S^W\). So

\[
P/Q = (U_1/Q)P_1 + (U_2/Q)P_2 + \cdots + (U_s/Q)P_s,
\]

where \(U_1/Q, U_2/Q, \ldots, U_s/Q \in \mathbb{C}(S^W)\).

Therefore \(P_1, P_2, \ldots, P_s\) is a spanning, linearly independent set of \(\mathbb{C}(V)\), i.e. a basis of \(\mathbb{C}(V)\). Thus \(s = |W|\) and \(\dim_{\mathbb{C}}(S_W) = |W|\).

\[\square\]

**References**

