Moment Hypergraphs for Homotopy Lie Groups

Abstract
Homotopy Lie groups are the homotopy analogues of compact Lie groups. We are interested in the torus equivariant cohomology of the homogeneous spaces of homotopy Lie groups. In this poster we show how the definitions and results from compact Lie group theory (in the yellow column) extend to definitions and results for homotopy Lie group theory (in the blue column) including the concept of moment graphs, some examples and a conjecture on a generalized GKM condition.

The geometric machinery of Lie group theory is not available for homotopy Lie groups. However, geometric intuitions in the homotopy case often make sense after translation to homotopic or combinatorial terms, an instance of which is displayed in the following.

Compact Lie Groups
Compact connected Lie groups are classified by their Weyl groups, which are crystallographic reflection groups. We have a 1-1 correspondence

\[
\text{Compact connected Lie groups} \longleftrightarrow \text{Crystallographic reflection groups} \subseteq \text{Real Lie groups}
\]

Example: Type $A_2$
The real reflection group $A_2 \subset GL(2, \mathbb{R})$ is abstractly isomorphic to the symmetric group $S_3$.
The group $A_2$ is generated by two reflections $s_1, s_2 \in A_2$ of order 2 with roots $\alpha, \beta \in \mathbb{R}[x_1, x_2]$ respectively.

Moment Graphs
Let $W \subset GL(r, \mathbb{R})$ be a crystallographic reflection group, and let $S = \bigoplus_{w \in \mathcal{W}} \mathbb{C}[x_1, \ldots, x_r]$. The moment graph $G_W$ of $W$ is given by:
- The set of vertices is $W$.
- For $w \in W$ and a reflection $s \in W$ with root $\alpha \in S$, the vertices $w$, $sw$ are joined by an edge labelled $\alpha$.

Associated to the moment graph $G_W$ we have the structure algebra $Z_W$ of $W$,
\[
Z_W = \left\{ (\psi) \in S \mid \alpha \text{ divides } \psi_w - \psi_{sw} \text{ for any edge } \alpha \right\}.
\]

Example: Type $A_2$

\[
\begin{align*}
G_{A_2} & : \quad \begin{array}{c}
\alpha + \beta \\
\alpha - \beta \\
\alpha + 3\beta \\
\alpha - 3\beta
\end{array} \\
\alpha + \beta & : \quad \begin{array}{c}
\alpha + \beta \\
\alpha - \beta \\
\alpha + 3\beta \\
\alpha - 3\beta
\end{array} \\
\alpha + 3\beta & : \quad \begin{array}{c}
\alpha + \beta \\
\alpha - \beta \\
\alpha + 3\beta \\
\alpha - 3\beta
\end{array} \\
\alpha - 3\beta & : \quad \begin{array}{c}
\alpha + \beta \\
\alpha - \beta \\
\alpha + 3\beta \\
\alpha - 3\beta
\end{array}
\end{align*}
\]

Theorem
Let $G$ be a compact connected Lie group with maximal torus $T$ and Weyl group $W$.
Then the $T$-equivariant cohomology of the homogeneous space $G/T$ is isomorphic to the structure algebra $Z_W$ of $W$,
\[
H^*_T(G/T) \cong Z_W.
\]
This theorem was proved by Goresky, Kottwitz and MacPherson in [GKM].

References

Definition
Let $p \in \mathbb{Z}$ be a prime. A homotopy Lie group (or $p$-compact group) is a pair $(X, BX)$ of spaces such that:
- $BX$ is $p$-complete (in the sense of [BK]).
- $H^*_T(X, F_p)$ is finite dimensional as a vector space over $F_p$.
- $X$ is homotopy equivalent to $\Omega BX$.

Homotopy Lie groups are the homotopy analogues of compact Lie groups. This means that homotopy Lie groups behave like compact Lie groups in many ways but are objects solely defined in homotopy theory, i.e. they do not have algebraic or geometric structure.

Homotopy Lie Groups
Homotopy Lie groups are classified in [AGMV] by their Weyl groups, which are $p$-adic reflection groups. We have a 1-1 correspondence

\[
\text{Connected homotopy Lie groups} \longleftrightarrow \text{p-Adic reflection groups} \subseteq \text{Complex reflection groups}
\]

Example: Type $G_{2,1,1}$
The complex reflection group $G_{2,1,1} \subset GL(1, \mathbb{C})$ is abstractly isomorphic to the cyclic group $\mathbb{Z}/3\mathbb{Z}$.
The group $G_{2,1,1}$ is generated by a single reflection $s \in G_{2,1,1}$ of order 3 with root $\alpha \in \mathbb{C}[x]$. $G_{2,1,1}$ is the Weyl group of the homotopy Lie group $S^5$, the 5-dimensional Sullivan sphere (the $p$-completion of the 5-dimensional sphere $S^5$).

Moment Hypergraphs
Let $W \subset GL(r,\mathbb{C})$ be a complex reflection group, and let $S = \bigoplus_{w \in \mathcal{W}} \mathbb{C}[x_1, \ldots, x_r]$. The moment hypergraph $G_W$ of $W$ is given by:
- The set of vertices is $W$.
- For $w \in W$ and a reflection $s \in W$ of order $n$ root $\alpha \in S$, the vertices $w, sw, s^2w, \ldots, s^{n-1}w$ in $G_W$ are all joined by a hyperedge labelled $\alpha$.

Associated to the moment graph $G_W$ we have the structure algebra $Z_W$ of $W$,
\[
Z_W = \left\{ (\psi) \in S \mid \alpha^n \text{ divides } \sum_{i=0}^{n-1} \mu_i^{(n-k)} \psi_{sw} \text{ for any hyperedge } \alpha \right\},
\]
where $\mu_i^{(n)} = e^{2\pi i/n} \in \mathbb{C}$ is a primitive $n^{th}$ root of unity.

Example: Type $G_{2,1,1}$

\[
\begin{align*}
G_{2,1,1} & : \quad \begin{array}{c}
\alpha \\
\beta
\end{array} \\
\alpha & : \quad \begin{array}{c}
\alpha \\
\beta
\end{array}
\end{align*}
\]

Conjecture
Let $X$ be a connected homotopy Lie group with maximal torus $T$ and Weyl group $W$.
Then the $T$-equivariant cohomology of the homogeneous space $X/T$ is isomorphic to the structure algebra $Z_W$ of $W$,
\[
H^*_T(X/T) \cong Z_W.
\]
This conjecture have been proven for cyclic groups and some cases of the family of groups $G_{m,n}$ so far.

Omar Ortiz
The University of Melbourne
ortizo@student.unimelb.edu.au
http://www.ms.unimelb.edu.au/~ortizo/

PhD Advisors:
Arun Ram
Craig Westerland