1. The Hopf algebras $\tilde{U}^{\geq 0}$ and $\tilde{U}^{\leq 0}$

1.1 Let $I$ be a finite set and let $Q = \sum_{i \in I} \mathbb{Z}$ be the free abelian group generated by the set $I$. Let $\mathbb{Z}$ be the free abelian group generated by the set $\mathbb{Z}$. Let $\langle , \rangle$ be a $\mathbb{Z}$ valued bilinear pairing between $Q$ and $Q$.

1.2 Let $\tilde{U}^{\geq 0}$ be the Hopf algebra over $Q(q)$ with generators

$$E_i, \quad i \in I, \quad K_\mu, \quad , \mu \in Q,$$

and relations

$$K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu}$$
$$K_\mu E_i = q^{(\mu,i)} E_i K_\mu, \quad i \in I, \mu \in Q$$

and with coproduct given by

$$\Delta(K_\mu) = K_\mu \otimes K_\mu,$$
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i.$$

1.3 Let $\tilde{U}^{\leq 0}$ be the Hopf algebra over $Q(q)$ with generators

$$F_i, \quad i \in I, \quad K_\mu^*, \quad , \mu \in Q,$$

and relations

$$K_0^* = 1, \quad K_\lambda K_\mu^* = K_{\lambda+\mu}$$
$$K_\mu^* F_i = q^{(-\mu,i)} F_i K_\mu^*, \quad i \in I, \mu \in Q$$

and with coproduct given by

$$\Delta(K_\mu^*) = K_\mu^* \otimes K_\mu^*,$$
$$\Delta(F_i) = F_i \otimes K_{-i}^* + 1 \otimes F_i.$$

1.4

$$\Delta(x) = \sum x^{(1)} K_{x^{(2)}} \otimes x^{(2)},$$
$$\Delta(y) = \sum y^{(1)} \otimes K_{-y^{(2)}}^{(1)} y^{(2)},$$

1.5

$$\Delta(E^{(p)}) = \sum_{t+t'=p} q^{t't'} E_i^{(t)} K_{t'i} \otimes E_i^{(t')},$$
$$\Delta(F^{(p)}) = \sum_{t+t'=p} q^{t't'} F_i^{(t)} K_{t'i} \otimes F_i^{(t')}.$$
1.4 We would like to identify $\tilde{U}^{\leq 0}$ with the Hopf algebra $(\tilde{U}^{\geq 0})^{*}_{\text{coop}}$.

(2.1) Proposition. There is a unique bilinear pairing

$$\tilde{U}^{\geq 0} \times \tilde{U}^{\leq 0} \to \mathbb{Q} \langle q \rangle$$

given by

$$\langle E_i, F_j \rangle = -\frac{\delta_{ij}}{q - q^{-1}},$$

$$\langle x, y_{1}y_{2} \rangle = \langle \Delta(x), y_{1} \otimes y_{2} \rangle$$

$$\langle x_{1} \otimes x_{2}, \Delta(y) \rangle = \langle x_{2}x_{2}, y \rangle.$$

Proof. The following relations are forced by the conditions.

1a) $\langle K_{\mu}, 1^* \rangle = 1$, $\langle E_{i}, 1^* \rangle$.

1b) $\langle 1, K_{\mu}^* \rangle = 1$, $\langle 1, F_{i} \rangle$.

1c) $\langle K_{\mu}, K_{\lambda}^* \rangle = \langle K_{\mu}, K_{\lambda}^{*|\mu} \rangle \neq 0$.

1d) $\langle E_{i}, K_{\mu}^* \rangle = 0$, $\langle K_{\mu}, F_{i} \rangle = 0$.

2) If $x \in \tilde{U}^{\geq 0}$ and $y \in \tilde{U}^{\leq 0}$ are homogeneous and $|x| \neq |y|$ then $\langle x, y \rangle = 0$.

3) If $x \in \tilde{U}^{\geq 0}$ and $y \in \tilde{U}^{\leq 0}$ then

$$\langle xK_{\mu}, y \rangle = \langle x, yK_{\mu}^* \rangle = \langle x, y \rangle.$$

4) If $x \in \tilde{U}^{\geq 0}$ and $y \in \tilde{U}^{\leq 0}$ then

$$\langle K_{\mu}x, y \rangle = \langle x, y \rangle \langle K_{\mu}, K_{-|x|}^* \rangle.$$

$$\langle x, K_{\mu}^*y \rangle = \langle x, y \rangle \langle K_{|x|}, K_{\mu}^* \rangle.$$

5) $\langle K_{\mu}, K_{\lambda}^* \rangle = q^{-\langle \lambda, \mu \rangle}$.

6) If $x, z \in \tilde{U}^{\geq 0}$ and $y, w \in \tilde{U}^{\leq 0}$ then

$$\langle xz, yw \rangle = \sum_{x, y, z, w} \langle x_{1}(1), y_{2}(1) \rangle \langle x_{2}(2), w_{2}(2) \rangle \langle z_{1}(1), y_{1}(1) \rangle \langle z_{2}(2), w_{1}(1) \rangle q^{-\langle |x_{1}|, |z_{1}| \rangle} q^{-\langle |x_{2}|, |z_{2}| \rangle} q^{\langle |x_{2}|, |w_{2}| \rangle}$$

3. The radical of the form $\langle \cdot, \cdot \rangle$.

3.1 Let $I^+$ and $I^-$ be the left and the right radicals of the form $\langle \cdot, \cdot \rangle$, respectively;

$$I^+ = \{ x \in \tilde{U}^{\geq 0} \mid \langle x, y \rangle = 0, \text{ for all } y \in \tilde{U}^{\leq 0} \}.$$

$$I^- = \{ y \in \tilde{U}^{\leq 0} \mid \langle x, y \rangle = 0, \text{ for all } x \in \tilde{U}^{\geq 0} \}.$$

Then

1) $I^+ \cap \tilde{U}^{0} = 0$.

2) $I^+ \text{ is a graded vector space.}$

3) $I^+$ is an ideal.

4) $I^+$ is a coideal.

Proof. 1) Since $\langle K_{\mu}, 1^* \rangle = 1$, $K_{\mu} \neq I^+$.

2) Suppose $x \in \tilde{U}^{\geq 0}$ and $x = \sum_{\nu \in Q^+} x_{\nu}$ where each $x_{\nu}$ is homogeneous and $|x_{\nu}| = \nu$. Fix $\mu \in Q^+$.

Then, the homogeneity of the form $\langle \cdot, \cdot \rangle$,

$$\langle x_{\mu}, z \rangle = \langle x_{\mu}, z_{\mu} \rangle = \langle x, z_{\mu} \rangle = 0$$

for all $z \in \tilde{U}^{\leq 0}$. Thus $x_{\mu} \in I^+$. Thus $I^+$ is a homogeneous ideal.
3) Assume \( x \in \tilde{U}_{\geq 0} \). Then, for every \( y \in \tilde{U}_{\geq 0} \) and every \( z \in \tilde{U}_{\leq 0} \),

\[
\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle \\
= \langle y \otimes x, \sum_z \tilde{z}_{(1)} \otimes K_{-|\mu|} \tilde{z}_{(2)} \rangle \\
= \sum_y \langle y, \tilde{z}_{(1)} \rangle \langle x, K_{-|\mu|} \tilde{z}_{(2)} \rangle \\
= 0.
\]

It follows that \( \mathcal{I}^+ \) is a right ideal of \( \tilde{U}_{\geq 0} \). The proof that \( \mathcal{I}^+ \) is a left ideal of \( \tilde{U}_{\geq 0} \).

4) First let us show that the left radical \( (\mathcal{I}^\otimes)^+ \) of the form

\[
(\cdot, \cdot): (\tilde{U}_{\geq 0} \otimes \tilde{U}_{\geq 0}) \times (\tilde{U}_{\leq 0} \otimes \tilde{U}_{\leq 0}) \rightarrow \mathbb{Q}(q)
\]

is \( \mathcal{I}^+ \otimes \tilde{U}_{\geq 0} + \tilde{U}_{\geq 0} \otimes \mathcal{I}^+ \). Let \( B \) be a basis of \( \tilde{U}_{\geq 0} \) consisting of homogeneous elements. Let

\[
\sum_{b, b'} c_{bb'} b \otimes b' \in (\mathcal{I}^\otimes)^+.
\]

Assume that \( b' \) is not in \( J^+ \). Since each homogeneous component of \( \tilde{U}_{\geq 0} \) is finite dimensional it follows that there is an element of \( \tilde{U}_{\geq 0} \) such that \( \langle z_{b'}, b \rangle = \delta_{b'b} \). Then

\[
0 = \langle \sum_{b, b'} c_{bb'} b \otimes b', z \otimes z_{b'} \rangle = \langle \sum_b c_{bb'} b, z \rangle
\]

for all \( z \in \tilde{U}_{\leq 0} \). It follows that \( \langle \sum_b c_{bb'} b, z \rangle \in \mathcal{I}^+ \). This argument shows

\[
(\mathcal{I}^\otimes)^+ \subseteq \mathcal{I}^+ \otimes \tilde{U}_{\geq 0} + \tilde{U}_{\geq 0} \otimes \mathcal{I}^+.
\]

The other inclusion is easy.

The fact that \( \mathcal{I} \) is a coideal now follows, since the equation

\[
\langle x, y_1 y_2 \rangle = \langle \Delta(x), y_1 \otimes y_2 \rangle
\]

implies that if \( x \in \mathcal{I}^+ \) then \( \Delta(x) \in (\mathcal{I}^\otimes)^+ \).

3.2 The quantum Serre relations are in the ideal \( \mathcal{I}^+ \).

Proof. \( \Box \)

4. The double

4.1 The following relations are determined by the definition of the multiplication in \( D(\tilde{U}_{\geq 0}) \):

\[
E_i F_j - F_j E_i = \frac{K_i - K_j^*}{q - q^{-1}} \\
K_\lambda K_\mu^* = K_\mu^* K_\lambda \\
K_\mu F_i = q^{-\langle \mu, i \rangle} F_i K_\mu \\
K_\mu^* E_i = q^{\langle \mu, i \rangle} E_i K_\mu^*
\]

4.2 Let \( \tilde{U} = D(\tilde{U}_{\geq 0})/J \) where \( J \) is the ideal generated by the relations \( K_\mu = K_\mu^* \). Clearly this ideal is also a coideal. Thus, \( \tilde{U} \) is a Hopf algebra with generators

\[
E_i, F_i, K_\mu
\]
which satisfy the relations

\[ K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu} \]
\[ K_\mu E_i = q^{(\mu,i)} E_i K_\mu, \quad i \in I, \mu \in Q \]
\[ K_\mu F_i = q^{(-\mu,i)} F_i K_\mu, \quad i \in I, \mu \in Q, \]
\[ E_i F_j - F_j E_i = K_i - K_{-i} \]

and has coproduct given by

\[ \Delta(K_\mu) = K_\mu \otimes K_\mu, \]
\[ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \]
\[ \Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i. \]

The triangular decomposition shows that this is a presentation of \( \tilde{U} \).

4.3 The map

\[ \Phi: \tilde{U} \otimes U^0 \rightarrow D(\tilde{U} \geq 0) \]
\[ K_\mu \otimes K_\nu \mapsto K_{\mu+\nu} K^\nu_{\mu}, \]
\[ E_i \otimes 1 \mapsto E_i, \]
\[ F_i \otimes 1 \mapsto F_i \]

is an isomorphism of algebras.