Free algebras $P$ and $F$

$P$, the free algebra generated by $p_1, \ldots, p_n$, and $F$, the free algebra generated by $f_1, \ldots, f_n$, are graded dual via $\langle, \rangle : P \times F \to C$ given by

$$\langle p_{i_1} \cdots p_{i_d}, f_{j_1} \cdots f_{j_e} \rangle = \delta_{d,e} \delta_{i_1,j_1} \cdots \delta_{i_d,j_e}$$

and

$$\deg(p_{i_1} \cdots p_{i_d}) = \deg(f_{j_1} \cdots f_{j_e}) = x_{i_1} + \cdots + x_{i_d}$$

is in

$$Q = \mathbb{Z}_2\text{-span}\{x_1, \ldots, x_r\}$$

$P$ has basis $\{p_{i_1}^{(n_1)} \cdots p_{i_m}^{(n_m)}\}$

$F$ has basis $\{f_{j_1}^{(n_1)} \cdots f_{j_m}^{(n_m)}\}$

Dual maps:

$$\begin{align*}
m &: P \otimes P \to P & \Delta &: F \to F \otimes F \\
r &: P \to P \otimes P & \circ &: F \otimes F \to F \\
- &: P \to P & - &: F \to F \\
\Delta_{\otimes} &: P \to P & \Delta &: F \to F
\end{align*}$$

This means

$$\langle x, y y' \rangle = \langle r(x), y y' \rangle \quad \text{and} \quad \langle x x', y \rangle = \langle x \Delta_{\otimes} x', \Delta_{\otimes} y \rangle$$
Dialgebra structure on $P$

$m : P \otimes P \to P$ given by $m(u \otimes v) = uv,$

$r : P \to P \otimes P$ given by $r(p_i) = p_i \otimes 1 + 1 \otimes p_i,$

where the product on $P \otimes P$ is given by

$\langle x \otimes y, z \otimes w \rangle = q^{-\langle \deg(y), \deg(z) \rangle} x \otimes y \otimes z \otimes w$

and

$r_i : P \to P$ by $r_i = p_i$ and $q = q^{-1}.$

and

$I_{p_i} : P \to P$ by

$I_{p_i}(x) = x p_i.$

Dialgebra structure on $F$

$o : F \otimes F \to F$ is given by

$u \ast v = \sum_{\sigma \in S_{k+l}} q^{\text{wt}(\sigma, uv)} \sigma(uv),$

where $k = \ell(u), l = \ell(v)$, the sum is over minimal length coset representatives of $S_k \times S_l$-cosets in $S_{k+l},$

$\text{wt}(\sigma, uv) = \sum_{1 \leq i, j \leq k+l} -\langle u_i, v_{k-j} \rangle,$

where $u_i$ is the $i$th letter in $u$ and

$v_{k-j}$ is the $(k-j)$th letter in $v.$
Δ: \( F \to F \otimes F \) is given by

\[
\Delta(u) = \sum_{v \otimes w \in F} \delta_{vw} \cdot v \otimes w
\]

and is a homomorphism with respect to the product

\[
(x \otimes y) \cdot (z \otimes w) = q^{-\langle \deg(y), \deg(z) \rangle} (x \otimes z) \otimes (y \otimes w).
\]

Define

\[
\gamma : F \to F \text{ by } \gamma_i = f_i \text{ and } \gamma = \gamma^{-1}
\]

and

\[
\Delta_{an} : F \to F \text{ by }
\]

\[
\Delta_{an}(x) = \sum_{u \in X} w, \quad u = wa^n
\]

so that the sum is over terms in \( x \) that end in \( an \).

Hence

\[
\Delta_{an}(f_1 \ldots f_d) = \begin{cases} f_1 \ldots f_{d-n}, & \text{if } j_{d-n+1} = \ldots = j_d = a, \\ 0, & \text{otherwise} \end{cases}
\]

**Key point**

\( U \hookrightarrow F \) and \( P \twoheadrightarrow U \)

**Note:** Make precise: "The same relations are the radical of \( \langle > \)."
The quantum group and categorification

Let $F$ be the vector space $F$ with product $\cdot : F \times F \to F$. Define a homomorphism

$\Phi : F \to \hat{F},$

$s_i \mapsto s_i,$

so that

$F \xrightarrow{\Phi} \hat{F}$

where $\hat{F} = F / \ker \Phi.$

As pointed out to me by H. Oguia, if the Cartan matrix is the $0$ matrix then

$F = T(V) = \text{tensor algebra},$

$\hat{F} = T^\vee(V) = \text{shuffle algebra},$

and

$T(V) \xrightarrow{g} T^\vee(V)$

by

$g(u_1 \cdots u_k) = \sum_{\sigma \in S_k} u_{\sigma(1)} \cdots u_{\sigma(k)},$

and $g$ is the unique map such that $g(\cdot \cdot \cdot \cdot v) = uv$ and $g(s_i) = s_i.$

Note that $g$ is an Hopf algebra map.