1 Column strict tableaux

A letter is an element of $B(\varepsilon_1) = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and a word of length $k$ is an element of $B(\varepsilon_1)^{\otimes k} = \{\varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$.

For $1 \leq i \leq n-1$ define

\[ \tilde{f}_i : B(\varepsilon_1)^{\otimes k} \rightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e}_i : B(\varepsilon_1)^{\otimes k} \rightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\} \]

as follows. For $b \in B(\varepsilon_1)^{\otimes k}$,

- place $+1$ under each $\varepsilon_i$ in $b$,
- place $-1$ under each $\varepsilon_{i+1}$ in $b$, and
- place $0$ under each $\varepsilon_j$, $j \neq i, i+1$.

Ignoring 0s, successively pair adjacent $(-1, +1)$ pairs to obtain a sequence of unpaired $+1$s and $-1$s

\[ +1 +1 +1 +1 +1 +1 +1 +1 -1 -1 -1 -1 \]

(after pairing and ignoring 0s). Then

\[ \tilde{f}_i b = \text{same as } b \text{ except the letter corresponding to the rightmost unpaired } +1 \text{ is changed to } \varepsilon_{i+1}, \]
\[ \tilde{e}_i b = \text{same as } b \text{ except the letter corresponding to the leftmost unpaired } -1 \text{ is changed to } \varepsilon_i. \]

If there is no unpaired $+1$ after pairing then $\tilde{f}_i b = 0$.
If there is no unpaired $-1$ after pairing then $\tilde{e}_i b = 0$.

A partition is a collection $\mu$ of boxes in a corner where the convention is that gravity goes up and to the left. As for matrices, the rows and columns of $\mu$ are indexed from top to bottom and left to right, respectively.

The parts of $\mu$ are $\mu_i = \text{(the number of boxes in row } i \text{ of } \mu)$,
the length of $\mu$ is $\ell(\mu) = \text{(the number of rows of } \mu)$,
the size of $\mu$ is $|\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} = \text{(the number of boxes of } \mu).$
Then $\mu$ is determined by (and identified with) the sequence $\mu = (\mu_1, \ldots, \mu_\ell)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$, where $\ell = \ell(\mu)$. For example,

$$(5, 5, 3, 3, 1, 1) = \begin{array} \text{?} \end{array}.$$ 

A partition of $k$ is a partition $\lambda$ with $k$ boxes. Write $\lambda \vdash k$ if $\lambda$ is a partition of $k$. Make the convention that $\lambda_i = 0$ if $i > \ell(\lambda)$. The dominance order is the partial order on the set of partitions of $k$,

$$P^+(k) = \{\text{partitions of } k \} = \{\lambda = (\lambda_1, \ldots, \lambda_\ell) \mid \lambda_1 \geq \cdots \geq \lambda_\ell > 0, \lambda_1 + \cdots + \lambda_\ell = k\},$$

given by

$$\lambda \geq \mu \quad \text{if} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for all } 1 \leq i \leq \max\{\ell(\lambda), \ell(\mu)\}.$$ 

For example, for $k = 6$ the Hasse diagram of the dominance order is

```
(16) —— (214) —— (2212)
    /   \         /   \         /   \\
   (313) —— (321) —— (42) —— (51) —— (6)
      \   /     \   /     \   /     \\
       (32) —— (42) —— (51) —— (6)
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Let $\lambda$ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ be a sequence of nonnegative integers. A column strict tableau of shape $\lambda$ and weight $\mu$ is a filling of the boxes of $\lambda$ with $\mu_1$ 1s, $\mu_2$ 2s, $\ldots$, $\mu_n$ ns, such that

(a) the rows are weakly increasing from left to right,
(b) the columns are strictly increasing from top to bottom.

If $p$ is a column strict tableau write $\text{shp}(p)$ and $\text{wt}(p)$ for the shape and the weight of $p$ so that

$$\text{shp}(p) = (\lambda_1, \ldots, \lambda_n), \quad \text{where } \lambda_i = \text{number of boxes in row } i \text{ of } p, \quad \text{and}$$
$$\text{wt}(p) = (\mu_1, \ldots, \mu_n), \quad \text{where } \mu_i = \text{number of } i \text{ s in } p.$$ 

For example,

$$p = \begin{array} {cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 & 3 & 3 & 4 & \\
3 & 3 & 3 & 4 & 4 & 4 & 5 & \\
4 & 5 & 5 & 6 & & & & \\
6 & 7 & & & & & & \\
7 & & & & & & & \\
\end{array}$$

has $\text{shp}(p) = (9, 7, 7, 4, 2, 1, 0)$ and $\text{wt}(p) = (7, 6, 5, 5, 3, 2, 2)$. 

2
For a partition $\lambda$ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$B(\lambda) = \{\text{column strict tableaux } p \mid \text{shp}(p) = \lambda\},$$
$$B(\lambda)_\mu = \{\text{column strict tableaux } p \mid \text{shp}(p) = \lambda \text{ and wt}(p) = \mu\},$$

(1.2)

Let $\lambda$ be a partition with $k$ boxes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$ 

The set $B(\lambda)$ is a subset of $B(\varepsilon_1)^{\otimes k}$ via the injection

$$B(\lambda) \hookrightarrow B(\varepsilon_1)^{\otimes k} \quad \text{p} \mapsto (\text{the arabic reading of } p)$$

where the arabic reading of $p$ is $\varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_k}$ if the entries of $p$ are $i_1, i_2, \ldots, i_k$ read right to left by rows with the rows read in sequence beginning with the first row.

**Proposition 1.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $k$ boxes. Then $B(\lambda)$ is the subset of $B(\varepsilon_1)^{\otimes k}$ generated by

$$p_\lambda = \varepsilon_{\lambda_1} \otimes \varepsilon_{\lambda_1} \cdots \otimes \varepsilon_{\lambda_1} \otimes \varepsilon_{\lambda_2} \otimes \varepsilon_{\lambda_2} \cdots \otimes \varepsilon_{\lambda_2} \otimes \cdots \otimes \varepsilon_{\lambda_n} \otimes \varepsilon_{\lambda_n} \otimes \cdots \otimes \varepsilon_{\lambda_n}$$

under the action of the operators $\tilde{e}_i, \tilde{f}_i, 1 \leq i \leq n$.

**Proof.** If $P = P(b)$ is a filling of the shape $\lambda$ then $b_{i_1} \otimes \cdots \otimes b_{i_k} = b$ is obtained from $P$ by reading the entries of $P$ in arabic reading order (right to left across rows and from top to bottom down the page). The tableau

$$P_\lambda = P(p_\lambda) =$$

is the column strict tableau of shape $\lambda$ with 1s in the first row, 2s in the second row, and so on. Define operators $\tilde{e}_i$ and $\tilde{f}_i$ on a filling of $\lambda$ by

$$\tilde{e}_i P = P(\tilde{e}_i p) \quad \text{and} \quad \tilde{f}_i P = P(\tilde{f}_i b), \quad \text{if } P = P(b).$$

To prove the proposition we shall show that if $P$ is a column strict tableau of shape $\lambda$ then

(a) $\tilde{e}_i P$ and $\tilde{f}_i P$ are column strict tableaux,
(b) $P$ can be obtained by applying a sequence of $\tilde{f}_i$ to $P\lambda$. Let $P^{(j)}$ be the column strict tableau formed by the entries of $P$ which are $\leq j$ and let $\lambda^{(j)} = \text{shp}(P^{(j)})$. Identify $P$ with the sequence

$$P = (\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda),$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip for each $1 \leq i \leq n$.

(a) Let us analyze the action of $\tilde{e}_i$ and $\tilde{f}_i$ on $P$. The sequence of $+1, -1, 0$ constructed in (???)

is given by

placing $+1$ in each box of $\lambda^{(i)}/\lambda^{(i-1)}$,
placing $-1$ in each box of $\lambda^{(i+1)}/\lambda^{(i)}$,
placing $0$ in each box of $\lambda^{(j)}/\lambda^{(j-1)}$, for $j \neq i, i+1$, and reading the resulting filling in Arabic reading order, see (???). The process of removing $+1, -1$ pairs can be executed on the horizontal strips $\lambda^{(i+1)}/\lambda^{(i)}$ and $\lambda^{(i)}/\lambda^{(i-1)}$,

$$\lambda^{(i+1)} = \lambda^{(i-1)}$$

with the effect that the entries in any configuration of boxes of the form

+1 +1 · · · +1
-1 -1 · · · -1

will be removed. Other $+1, -1$ pairs will also be removed and the final sequence

$$-1 -1 \cdots -1 +1 +1 \cdots +1$$

(1.3)

will correspond to a configuration of the form

$$\lambda^{(i+1)} = \lambda^{(i-1)}$$

The rightmost $-1$ in the sequence (*) corresponds to a box in $\lambda^{(i+1)}/\lambda^{(i)}$ which is leftmost in its row and which does not cover a box of $\lambda^{(i)}/\lambda^{(i-1)}$. Similarly the leftmost $+1$ in the sequence (*) corresponds to a box in $\lambda^{(i)}/\lambda^{(i-1)}$ which is rightmost in its row and which does not have a box of $\lambda^{(i+1)}/\lambda^{(i)}$ covering it. These conditions guarantee that $\tilde{e}_iP$ and $\tilde{f}_iP$ are column strict tableaux.

(b) Applying the operator

$$\tilde{f}_{n,i} = \tilde{f}_{n-1} \cdots \tilde{f}_{i+1} \tilde{f}_i$$

to $P\lambda$
will change the rightmost $i$ in row $i$ to $n$. A sequence of applications of

$$\tilde{f}_{n,i}, \text{ as } i \text{ decreases (weakly) from } n - 1 \text{ to } 1,$$

can be used to produce a column strict tableau $P_n$ in which

1. the entries equal to $n$ match the entries equal to $n$ in $P$, and
2. the subtableau of $P_n$ containing the entries $\leq n - 1$ is $P_{\lambda(n-1)}$.

Iterating the process and applying a sequence of operators

$$\tilde{f}_{n-1,i}, \text{ as } i \text{ decreases (weakly) from } n - 2 \text{ to } 1,$$

to the tableau $P_n$ can be used to produce a tableau $P_{n-1}$ in which

1. the entries equal to $n$ and $n - 1$ match the entries equal to $n$ and $n - 1$ in $P$, and
2. the subtableau of $P_{n-1}$ containing the entries $\leq n - 2$ is $P_{\lambda(n-2)}$.

The tableau $P$ is obtained after a total of $n$ iterations of this process. □