G = a complex connected reductive algebraic group
B = a Borel subgroup
T = a maximal torus

Equivalent data:
W = finite reflection group
C = a fixed fundamental chamber
P = a W-invariant lattice.

Example 1:
\( G = GL_n(C) \)
\( B = \{ (I, d) \} \)
\( T = \{ (0, d) \} \)

\( W = S_n \) acting on \( \mathbb{R}^n = \bigoplus_{i=1}^{n} \mathbb{R} \epsilon_i \)

Reflections: \( s_{ij}, \) transposes \( \xi_i \) and \( \xi_j \)

\( C = \{ \lambda = (\lambda_1, 2\lambda_2, \ldots, n\lambda_n) \} \)

\( P = \mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z} \epsilon_i \)
Theorem (a) There is a bijection

\[ \{ \text{finite dimensional} \} \leftrightarrow P \]

\[ \text{simple } T \text{-modules} \]

\[ x^\mu : T \to \mathbb{C} \leftrightarrow 1^\mu \]

(b) There is a bijection

\[ P^+ \leftrightarrow \{ \text{finite dimensional} \} \]

\[ \text{simple } G \text{-modules} \]

\[ \lambda \mapsto L(\lambda) \]

where \( P^+ = P \cap \overline{C} \), \( \overline{C} \) is the closure of \( C \).

Let \( L(\lambda) \) be a simple \( G \)-module.

\[ \text{Res}_T^G (L(\lambda)) = \bigoplus_{\mu \in P} L(\lambda)_\mu, \text{ where} \]

\[ L(\lambda)_\mu = \{ m \in L(\lambda) \mid tm = x^\mu(t)m, \text{ for } t \in T \} \]

The character of \( L(\lambda) \) is

\[ s_\lambda = \sum_\mu \dim (L(\lambda)_\mu) x^\mu \]

an element of \( \mathbb{C}[P] = \text{span}_\mathbb{C} \{ x^\mu \mid \mu \in P \} \) with \( x^\lambda x^\mu = x^{\lambda+\mu} \).

Goal: The crystal is an index set

\[ \hat{\mathcal{L}}(\lambda) = \bigcup_{\mu} \hat{\mathcal{L}}(\lambda)_\mu \leftrightarrow \text{basis of } L(\lambda) = \bigoplus_{\mu} L(\lambda)_\mu \]

such that

\[ s_\lambda = \sum_{\rho \in \hat{\mathcal{L}}(\lambda)} x^{\text{wt}(\rho)}, \text{ where } \text{wt}(\rho) = \mu \text{ if } \rho \in \hat{\mathcal{L}}(\lambda)_\mu. \]
The affine Hecke algebra

Let $H_1, \ldots, H_n$ be the walls of $C$ and the reflection in $H_i$.

The positive side of $H_i$ is the side towards $C$.

The affine Hecke algebra $H$ is given by generators $X^i, X \in P$ and $T_w, w \in W$ and relations

\[ X^i X^j = X^j X^i = X^j X^i, \]

\[ T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } s_i w > w, \\ q^{-2} T_{s_i w} + (1 - q^{-2}) T_w, & \text{if } s_i w < w. \end{cases} \]

If $\lambda$ is on the positive side of $H_i$, then

\[ X^i T_{s_i} = T_{s_i} X^i + (1 - q^{-2}) (X^{s_i \lambda} X^{i+} + \ldots + X^{\lambda-i-} + X^i) \]

Problem: Find $C_{\lambda w}^\mu$ such that

\[ X^i T_w = \sum_{\nu, \mu} C_{\lambda w}^\mu T_{\nu} X^\mu. \]

Idea: The crystal is the solution to this problem at $q = 0$. 
The Path model

$\text{wt}(p)$ = endpoint of $p$
$\tau(p)$ = initial direction of $p$
$\sigma(p)$ = final direction of $p$

Root operators

and define $\tau_i$ by

$$\tau_i F.p = p \quad \text{if } \tau_i p \neq 0.$$ 

A crystal is a set of paths closed under $\tau_i, \tilde{\tau}_i$.

Let

$$\hat{\mathcal{C}}(\lambda) = \text{crystal generated by } p_{\lambda}$$

Theorem

$$s_{\lambda} = \sum_{p \in \hat{\mathcal{C}}(\lambda)} x^{\text{wt}(p)}$$
Theorem (Pittie-Ram) let \( \varphi = 0 \) in \( \mathcal{X} \). Let \( x \in P^* \) and \( w \in W \). Then

\[
x^1 T_{w^{-1}} = \sum_{p \in Z(u)} T_{\varphi(p)+1} x^{w_t(p)}
\]

Example \( \xi(p) \) where \( \rho = \omega_1 + \omega_2 = \xi_1 + \xi_2 = \Xi_1 \Xi_2 \)

The crystal generated by \( p \).

The crystal generated by \( \xi(p) \).
Branching/Littlewood-Richardson rules

A highest weight path is a path \( p \leq c \cdot p \).

Theorem. Let \( B \) be a crystal. Then

\[
\text{char } B = \sum_{p \in B} s_{\mu}(p)_{p \leq c \cdot p}
\]

Example. Highest weight paths in \( \hat{\mathfrak{h}}(p) \otimes \hat{\mathfrak{h}}(p) \)

\[
\sum_{(1)} = s_0 + 2s_p + s_{-p} + s_{\pm p} + s
\]
Example: Highest weight paths in $\mathfrak{g} \text{ on}$

$\mathfrak{g} \text{ on}$

$\mathfrak{g} \text{ on}$

These paths give us a tower.