$P = \mathbb{Z}$-lattice

$W = a$ finite group acting on $P$

$C = a$ fixed fundamental chamber in $\mathbb{R}^n = \mathbb{R} \otimes \mathbb{Z} P$.

The quantum group associated to $(W, C, P)$ is the $\mathfrak{q}(q)$ algebra

$U = U_0 \cup U_{>0}$

$U_0$ is generated by $F_1, F_2, \ldots, F_n$

$U_{>0}$ is generated by $E_1, E_2, \ldots, E_n$

$U_0 = \mathbb{C}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$

with relations ... .

The Verma module is the $U$-module

$M(\lambda) = U_0 V_\lambda^+$ with $E_i V_\lambda^+ = 0$, $K_i V_\lambda^+ = q^{\mu(\lambda)} V_\lambda^+$. 

Algebra seminar, Verma crystals, University of Lyon
Fundamental data
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$U(\mathfrak{a})$ is the unique simple quotient of $H(\mathfrak{a})$.

Let $[k, 1] = \frac{q^{k} - q^{-k}}{q - q^{-1}}$ and $[m, 1] = [m, [m, 1]] \cdots [2, 1]$

Fix a reduced word $w_0 = s_{i_1} \cdots s_{i_N}$.

The negative root vectors are

\[ F_{i_1} = F_i, \quad F_{i_2} = F_{i_1} F_{i_2}, \quad \cdots, \quad F_{i_N} = T_{i_1} \cdots T_{i_{N-1}} F_{i_N} \]

where $T_i : W \to W$ are Lusztig's braid group automorphisms.

Recall:

\[ \beta_1 = e_i, \quad \beta_2 = s_i e_i, \quad \cdots, \quad \beta_N = s_i \cdots s_{i_{N-1}} e_i \]

are the positive roots.

The PBW bases of $U_{\mathfrak{a}}$ is

\[ \{ F^{(\vec{m})} = F_{\beta_1}^{(m_1)} \cdots F_{\beta_N}^{(m_N)} \mid \vec{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_{\geq 0}^N \} \]

Where $F_{\beta}^{(m)} = F_{\beta}^{m} = \frac{[m, 1]}{[m, 1]}$

The star involution is the $Q$-automorphism of $U$ given by

\[ \bar{e}_i = e_i, \quad \bar{F}_i = F_i, \quad \bar{K}_i = K_i^{-1}, \quad \bar{e} = e^{-1} \]
The canonical basis is

\[ B = \{ \delta^{(\hat{m})} \mid \hat{m} \in \mathbb{Z}^n_{\geq 0} \} \]

given by

1. \( \delta^{(\hat{m})} = \delta^{(\hat{m})} \)

2. \( \delta^{(\hat{m})} = \lambda^{(\hat{m})} + \sum_{\hat{m} \in \hat{m}^*} \rho_{\hat{m}, \hat{m}} F^{(\hat{m})} \text{ with } \rho_{\hat{m}, \hat{m}} \in \mathbb{Q}_{\geq 0} \).

\[ \lambda^{(\hat{m})} \mid \hat{m} \in \mathbb{Z}^n_{\geq 0} \] depends on the choice \( w_0 = s_{i_1} \cdots s_{i_n} \).

**Theorem (Losztn)**

(a) \( B \) does not depend on the choice \( w_0 = s_{i_1} \cdots s_{i_n} \)

(b) \( B(\lambda) = \{ \delta^{(\hat{m})} \mid \lambda^{(\hat{m})} \neq 0 \text{ in } \mathcal{L}(\lambda) \} \text{ is a basis of } \mathcal{L}(\lambda) \).

The Weyl polytope is the convex hull of \( \{ \delta^{(w)} \mid w \in W \} \).

\[ \mathcal{W}(\lambda) = \]
let $b \in B$. The $HV$ polytope of $d$ is

$$\left[ d \right] = \text{convex hull of } \{ dw \mid w \in W \}$$

where

$$d s_i, s_j = -m_i, -m_j, \ldots, -m_i' \mid s_i'. $$

Then

$$B \leftrightarrow \{ HV \text{polytopes} \}$$

$$d \mapsto \left[ d \right] = \begin{array}{c}
\text{Diagram}
\end{array}$$

and

$$B \Lambda \leftrightarrow \{ HV \text{polytopes} \ \left[ \bar{d} \right] \mid \left[ \bar{d} \right] + \lambda \subseteq \left[ \Lambda \right] \}.$$ 

Kamnitzer has described the crystal structure (root operators) on $HV$ polytopes to produce a Verma crystal.

Path crystals

\begin{align*}
&H_4, \\
&\text{Diagram 1} \\
&H_4, \\
&w t(p)
\end{align*}
Root operators

\[ f_{ij} p = p \text{ if } f_{ij} \neq 0. \]

A crystal is a set of paths which is closed under the root operators.

\[ B(p) \text{ is the crystal generated by } p. \]

\[ B(p) \rightarrow B(1000p) \]

\[ p \rightarrow p \otimes p. \]
Let \( C(A) = B(\infty A) \)

View \( \rho A \in C(A) \) as

\[ \rho A : \mathbb{R}_{>0} \to \mathbb{R}^n = \mathbb{R}^n \text{ with } \rho(1) = 1. \]

If \( \rho \in C(A) \) and

\[ t = (a, w_1, w_2) = a \underbrace{w_1 \ldots w_2}_{\text{a term of } \rho} \]

then let

\( u \) be the unique element (up to constant) in \( U_{\leq 0} \) s.t.

\[ H/w_1(a\rho - \rho) \to H/w_1(a\rho - \rho) \]

\[ v^+_w(a\rho - \rho) = u \] \( v^+_w(a\rho - \rho) \]

Let

\[ u = \frac{TT^* u}{\text{trace}} \]

Theorem (Hittelman):

\[ \{ u \rho \mid \rho \in C(A) \} \] is a basis of \( U_{\leq 0} \)
Theorem. Let \( \Lambda \in P^+ \) and \( \nu \in W \). Define 

\[
C(\nu \omega_\Lambda) = \{ p \in C(\Lambda) \mid \text{the \textit{tit} of } p \text{ is } \nu \omega_\Lambda \} \\
B(\nu \omega_\Lambda) = \{ p \in C(\nu \omega_\Lambda) \mid \text{the \textit{tail} of } p \text{ is } \Lambda \text{ straight} \} \\
C(\nu \omega_\Lambda)^+ = \{ p \in C(\nu \omega_\Lambda) \mid p \text{ is } \nu \text{-highest \textit{weight}} \} 
\]

Then

1. \( C(\nu \omega_\Lambda) \) is a model for \( M(\nu \omega_\Lambda) \) \\
   \( B(\nu \omega_\Lambda) \) is a model for \( L(\nu \omega_\Lambda) \)

2. (BGG resolution) There is an exact sequence of crystals 
   \[
   0 \to C(\nu \omega_\Lambda) \to \cdots \to \oplus C(\nu \omega_\Lambda) \to \cdots \to C(\Lambda) \to B(\Lambda) \to 0 
   \]

3. Let \( \mu \in P^+ \). The \( \gamma \)-weight multiplicity is

   \[
   K_\mu(\nu) = \sum_{p \in B(\Lambda) / \mu} \gamma(p) 
   \]

   where \( B(\Lambda) / \mu = \{ p \in B(\Lambda) \mid wt(p) = \nu \omega_\Lambda \text{ and } \gamma(p) \text{ is the depth of } p \} \)

4. Let \( \nu \in W \). The Kazhdan-Lusztig polynomial is

   \[
   P_{\nu \omega_\Lambda}(t) = \sum_{p \in C(\nu \omega_\Lambda)^+} t^d(p) 
   \]

   where \( d(p) \) is the depth of \( p \).