Why I care about $p$-compact groups

1. Symmetric functions (for $(S_n, \mathbb{Z}^n)$)
   3 formulas for the Schur function.
2. Symmetric functions (for $(W_0, \mathbb{Z}_p^*)$)
   3 formulas for the Weyl character.
3. The Chevalley classification
   The Borel-Weil-Bott formula.
4. Flag varieties: $K(GL)$ and $H^*(GL)$
   Pieri-Chevalley formulas.
5. The classification of $p$-compact groups.
   The Clark-Ewing formula.

Alternate title of this talk:
My life in Mathematics

Point of the talk:
The $p$-compact group corresp. to $(W_0, \mathbb{Z}_p^*)$
is the set of Littelmann paths corresp. to $(W_0, \mathbb{Z}_p^*)$. 
Symmetric functions for \((S_n, \mathbb{Z}^n)\) (circa 1988).

\(S_n\) acts on \(\mathbb{Z} [x_1, \ldots, x_n]\) by permuting \(x_1, \ldots, x_n\).

The ring of symmetric functions is

\[\mathbb{Z} [x_1, \ldots, x_n]^{S_n} = \left\{ f \in \mathbb{Z} [x_1, \ldots, x_n] \mid \forall \sigma \in S_n, f(\sigma(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n) \right\}\]

A partition with \(\leq n\) rows is a collection of boxes in a corner

\[
\lambda = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} = (4, 4, 2, 1, 1)
\]

\(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) with \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\)

where \(\lambda_i\) is \# of boxes in row \(i\).

Theorem: \(\mathbb{Z} [x_1, \ldots, x_n]^{S_n}\) has basis

\[
\{ s_\lambda \mid \lambda \text{ is a partition with } \leq n \text{ rows} \}
\]

where

\[
s_\lambda = \frac{\det(x_i^{\lambda_j + n-j})}{\det(x_i^{n-j})}
\]
3. Formulas for the Schur function

\[ S_n = \sum_{w \in S_n} \det(w) W(x_1^{\lambda_1 + n-1}, x_2^{\lambda_2 + n-2}, \ldots, x_n^{\lambda_n}) \]

\[ = \sum_{w \in S_n} \det(w) W(x_1^{n-1}, x_2^{n-2}, \ldots, x_{n-1}^{1}, x_n^{0}) \]

\[ = \sum_{T \text{ column strict}} x^{\operatorname{wt}(T)} \]

\[ = \chi((x_1^0, \ldots, 0^0), L(A)) \]

where

A column strict tableau \( T \) of shape \( \lambda \) filled from \( 1, 2, \ldots, n \) is a filling of the boxes of \( \lambda \) such that

(a) rows increase weakly (left to right),
(b) columns increase strictly (top to bottom)

and

\[ x^{\operatorname{wt}(T)} = x_1^{1's} \cdot x_2^{2's} \cdot \ldots \cdot x_n^{n's} \cdot m^T \]

and

\( L(A) \) is the finite dimensional \( \text{GL}_n(A) \) module

with a vector \( v \in L(A) \) such that

\[ \begin{pmatrix} x_1^k \vline \vline \vline x_n \end{pmatrix} v = x_1^k x_2^k \ldots x_n^k v. \]
Symmetric Functions for $(W_0, \mathfrak{g}_2^*)$

A Weyl group is a $\mathbb{Z}$-reflection group.

A $\mathbb{Z}$-reflection group is a pair $(W_0, \mathfrak{g}_2^*)$ where

$\mathfrak{g}_2^*$ is a representation of $W_0$ (a $\mathbb{Z}W_0$-module)

$W_0$ is a finite group

such that

$W_0 \leq GL(\mathfrak{g}_2^*) = GL_n(\mathbb{Z}) \leq GL_n(\mathbb{Q})$

is generated by reflections.

(ie reflection is a matrix with all but one eigenvalue equal to 1)

The group algebra of $\mathfrak{g}_2^*$ is

$\mathbb{C}[\mathfrak{g}_2^*] = \text{span}\{x^\lambda | \lambda \in \mathfrak{g}_2^*\}$ with $x^\lambda x^\mu = x^{\lambda + \mu}$

has a $W_0$-action given by

$w x^\lambda = x^{w \lambda}$, for $w \in W_0$, $\lambda \in \mathfrak{g}_2^*$.

$W_0$ acts on $\mathbb{R}^n = \mathbb{R} \otimes \mathfrak{g}_2^* = \mathbb{R}\text{-span} \{w_1, \ldots, w_n\}$.

Let $C$ be a fundamental region for $W_0$ acting on $\mathbb{R}^n$

$(\mathfrak{g}_2^*)^+ = \mathfrak{g}_2^* \cap C$ and $(\mathfrak{g}_2^*)^{++} = \mathfrak{g}_2^* \cap C$
Three formulas for the Weyl character.

The ring of symmetric functions is

\[ \mathbb{C}[x]^{W_0} = \{ f \in \mathbb{C}[x] \mid w f = f \text{ for } w \in W_0 \} \]

Let \( p \in \mathfrak{h}^* \) be such that

\[ (\mathfrak{h}^*)^+ \rightarrow (\mathfrak{h}^*)^+ \]

\[ \lambda \mapsto \lambda + p \]

is a bijection.

Then

\[ \mathbb{C}[x]^{W_0} \text{ has basis } \{ s_{\lambda} \mid \lambda \in (\mathfrak{h}^*)^+ \} \]

where

\[ s_{\lambda} = \sum_{w \in W_0} \text{det}(w) \, x^{ \lambda + p } \]

\[ = \sum_{w \in W_0} \text{det}(w) \, x^{ \lambda } \]

\[ = \sum_{p \in B(\mathfrak{h})} x^{\text{wt}(p)} \]

\[ = \text{Tr}(\cdot, \mathbb{C}(\lambda)) \]

where \( G \) is a compact Lie reductive complex algebraic group

corresponding to \( (W_0, \mathfrak{h}^*) \).

\( \mathbb{C}(\lambda) \) is an irreducible \( G \)-module.
and $\text{Tr}(. , L(\mathfrak{l})) : T \to C$

$t \mapsto \text{Tr}(t , L(\mathfrak{l}))$

where $T$ is a maximal torus of $G$.

Chevalley's Classification

There is an equivalence of categories

\[
\begin{align*}
\{ & \text{complex reductive algebraic groups } G \\ & \text{groups } (W_0, \mathfrak{Z}_2) \}
\end{align*}
\]

\[
\begin{align*}
\overset{\text{Borel choice of } B}{G} & \quad \overset{\text{maximal } T}{\leftarrow} \quad \overset{\text{choice of } C}{(W_0, \mathfrak{Z}_2)}
\end{align*}
\]

The flag variety is $G/B$.

Weyl's theorem - The category of skew representations of $G$ is a categorification of $\text{CE}(W_0)$ for which $\mathfrak{Z}_2$ corresponds to $L(\mathfrak{l})$.

The Borel-Weil-Bott formula

\[ H^0(G/B, L_\lambda) \cong L(\mathfrak{l}) \]

where $L_\lambda = G \times B C\nu$, where $C\nu$ is the 1-dimensional $B$-module given by $d\nu = X^H(b)\nu$, for $d \in B.$
The flag variety

\[ G = U / B w B \quad \text{for } w \in W_0. \]

The Schubert varieties are

\[ X_w = B w B \quad \text{in } G / B, \quad \text{for } w \in W_0. \]

Thus

\[ \{ X_w \mid w \in W_0 \} \] is a basis of \( H^* (G / B) \)

\[ \{ C_{wJ} \mid w \in W_0 \} \] is a basis of \( K (G / B) \).

The map

\[ \mathcal{O} X J \rightarrow K (G / B) \]

\[ X^1 \rightarrow [G x B, C_{wJ}] \]

is surjective, with kernel \( \mathcal{O} X J W_0 \).

The map

\[ S (1, \lambda^*) \rightarrow H^* (G / B) \]

\[ \lambda \rightarrow c_1 (L^\lambda) \]

is surjective, with kernel \( S (1, \lambda^*) W_0 \).

So

\[ K (G / B) = \mathcal{O} X J / \mathcal{O} X J W_0 \quad \text{and} \quad H^* (G / B) = S (1, \lambda^*) / S (1, \lambda^*) W_0. \]
Chevalley, about 1954, gave a formula: In $H^*(G/B)$

$$c_i(x_\lambda) \cdot [x_W] = \sum_{v \in W_0} c^v_{\lambda, w} [x_v],$$

where $c^v_{\lambda, w} = \ldots$

H. Pittie and I showed, in $K(G/B)$

$$[x_\lambda] \cdot [x_W] = \sum_{v \in W_0} d^v_{\lambda, w} [x_v],$$

where $d^v_{\lambda, w} = \#$ of paths $p \in P(\iota)$ with initial
direction $\lambda w$ and final direction $v$.

So it seems possible to understand everything
about $H^*(G/B)$ and $K(G/B)$
purely from the knowledge of $(W_0, \beta^*)$. 