References

(1) Atiyah and Bott, The Moment map and equivariant cohomology, Topology 23 (1984), 1-28

(2) Duistermaat and Heckman, On the Variation in the cohomology in the symplectic form of the reduced phase space, Invent. Math 69 (1982) 259-268


(4) Brion, Points entiers dans les polytopes convexes, Séminaire Bourbaki, 1993-94 n° 780.
Equivariant cohomology

Let \( T \) be a group and let \( ET \) be a contractible space on which \( T \) acts freely.

The **Borel construction** is the functor?

\[
\begin{align*}
\{ T\text{-spaces}\} & \longrightarrow \{ \text{fibre bundles}\} \\
on BT & \longmapsto ET \times_T M
\end{align*}
\]

Where \( BT = ET \times pt \) and \( ET \times_T M = \frac{ET \times M}{(xt, m) = (x, tm)} \).

**Remark:** If \( T \) is a compact Lie group acting smoothly on \( M \) then \( ET \times T M \) is homotopy equivalent to \( M/T \).

The equivariant cohomology is

\[
H^*_T : \{ T\text{-spaces}\} \longrightarrow \{ H^*_T(pt) \text{-modules}\}
\]

\[
M \longmapsto H^*_T(M) = H^*(ET \times_T M)
\]

**Remarks:**
(a) If \( T = (S^1)^n \) then \( H^*_T(pt) = \mathbb{C}[x_1, \ldots, x_n] \)

(b) If \( G \) is a compact Lie group then

\[
H^*_G(pt) = \mathbb{C}[x_1, \ldots, x_n] \cong H^*_T(pt)^W
\]

and

\[
H^*_G(M) = H^*_T(M)^W
\]

where \( T \) is a maximal torus of \( G \) and \( W \) is the Weyl group of \( G \).
Let $M$ be a $T$-space. If $M$ is nice then

$$K_T(M) = \text{Grothendieck group of } T\text{-equivariant vector bundles on } M$$

$$= \text{Grothendieck group of } T\text{-equivariant coherent sheaves on } M.$$

The point: If you allow yourself denominators the Chern character gives an isomorphism

$$\text{ch} : K_T(M) \to H^*_T(M)^\hat{}$$

where $H^*_T(M)^\hat{}$ is a completion of $H^*_T(M)$

Examples: (1) $K_T(pt) = \text{Grothendieck group of } T\text{-modules}$

$$K(pt) = K^*_T(pt) = \text{Grothendieck group of vector spaces}$$

(2) If $T = (\mathbb{C}^*)^n$ then $K_T(pt) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

We have isomorphisms: If $T$ is a subgroup of $G$ then

$$H^*_T(M) \cong H^*_G(G/TM) \text{ and } K_T(M) \cong K_G(G/TM).$$
**Pushforwards**

In $K$-theory:

(a) If $f: \mathcal{M} \to \mathcal{N}$ is a proper $\Gamma$-equivariant morphism then there is a morphism

\[ f_* : K_\Gamma(\mathcal{M}) \to K_\Gamma(\mathcal{N}) \]

(b) If $f: \mathcal{M} \to \mathcal{N}$ is $\Gamma$-equiv. a morphism then there is a pullback morphism

\[ f^* : K_\Gamma(\mathcal{N}) \to K_\Gamma(\mathcal{M}) \]

The **Umkehr homomorphism**: If $f: \mathcal{M} \to \mathcal{N}$ is a proper $\Gamma$-equivariant map there is a pushforward

\[ f_* : H^*_\Gamma(\mathcal{M}) \to H^*_\Gamma(\dim \mathcal{N} - \dim \mathcal{M})(\mathcal{N}) \]

which satisfies:

\[ (f \circ g)_* = f_* \circ g_* , \quad f_* (v \cdot f^*(u)) = (f_*) v u \]

and

If $f: \mathcal{M} \to \mathcal{N}$ is a fibration then

$f_*$ corresponds to integration over the fibre.
Localization Riemann-Roch

Let $f: X \to Y$ be a morphism. Then

\[ ch(f^*(E)) = f^*(ch(E)) \text{ and } f^*(\text{Todd}_X \cdot ch(E)) = \text{Todd}_Y \cdot ch(f^*(E)). \]

\[ \begin{array}{ccc}
K^*_f(X) & \xrightarrow{f^*} & K^*_f(Y) \\
ch \downarrow & & \downarrow ch \\
H^*_T(X) & \xrightarrow{f^*_T} & H^*_T(Y)
\end{array} \]

\[ \begin{array}{ccc}
K^*_T(X) & \xrightarrow{f^*_T} & K^*_T(Y) \\
\text{Todd}_X \cdot ch \downarrow & & \downarrow \text{Todd}_Y \cdot ch \\
H^*_T(X) & \xrightarrow{f^*_T} & H^*_T(Y)
\end{array} \]
Thom isomorphism

If \( f : N \hookrightarrow M \) is an inclusion of manifolds and \( \nu_N \) is the normal bundle to \( N \) in \( M \) then

\[
\tilde{f}_* : H^k(M, \mathbb{R}) \to H^k(M, \mathbb{R})
\]

The Thom class of \( \nu_N \) is \( \nu_N \).

and the Euler class of \( \nu_N \) is \( \frac{1}{f} f^+ \).

The most important pushforward is

\[
\text{Th}_* : H^*_e(M) \to H^*_e(pt)
\]
coming from \( M \to pt \).

This is the equivariant Euler characteristic of \( M \).

It corresponds to integrating over the fiber in
the fibration \( E T_e \to H \)

\[
BT = E T_e \to pt.
\]
Localization

Let $E$ be a vector bundle with a $T$-action, i.e. $T$ acts on $E$, and on each fiber by a linear action.

*See [CG] 3.5.11 and (5.11.4) and Cor. 6.1.17.

Integration formula: If $\pi: M \to pt$ then

$$\pi_*: H^*_T(M) \to H^*_{\pi}(pt)$$ is given by

$$\pi_* [\varphi] = \sum_{P} \pi_P^* \left( \frac{N_P^*}{E(P)} \right)$$

where the sum is over the connected components $P$ of $M$, $\pi_P: P \to M$, and

$$E(P) = \bigoplus_{j \in \mathbb{N}} N_j^P,$$

where $j$ are the indices if

$$N_j^P = \bigoplus_{j \in \mathbb{N}} X_j$$

as a $T$-module.

Here $N_j^P$ is the normal bundle to $\pi: M \to pt$.

In the notation of [CG] line after (5.11.4)

$$H^*_M = \bigoplus_j N_j$$ is the weight decomposition of the normal bundle, $\lambda = \bigoplus_i (\sum (-\lambda_i(j))^2 \chi_i N_{\lambda_i})$. 
Counting points in polytopes

A toric variety is a normal variety with a T-action with a dense orbit.

There is a bijection

\[
\begin{aligned}
\{ \text{integer polytopes} \} & \rightarrow \{ \text{pairs } (X,L) \text{ where } & \\
& X \text{ is a toric variety} \} \\
& L \text{ is an ample line bundle on } X
\end{aligned}
\]

An integer polytope is the convex hull of a finite number of points of \( \mathbb{Z}^n \) in \( \mathbb{R}^n \).

**Theorem (Erhart)** There is a polynomial

\[
\varphi(P) = a_0(P) + a_1(P) t + \ldots + a_n(P) t^n
\]

such that

\[
\varphi(kP) = \text{Card}(\mathbb{Z}^n \cap kP) \text{ for } k \in \mathbb{Z}^+.
\]

**Theorem**

\[
2\varphi(k) = \chi(X, L^{\otimes k})
\]
Moment maps

A action on $M$ preserve a symplectic form $\omega$ on $M$. $\omega \in \Omega^2(M)$ is closed and

\[ \frac{\omega^n}{n!} \]

is nowhere $0$ on $M$.

Define

\[ \mu : \mathfrak{g} \to \text{Lie-invariant vector fields on } M. \]

The moment map is

\[ \mathbf{\Phi} : M \to \mathfrak{g}^\ast. \]

Then the support

(a) $\mathbf{\Phi}(\frac{\omega^n}{n!})$ is a convex polytope.

(b) the measure $\mathbf{\Phi}(\frac{\omega^n}{n!})$ is a piecewise-polynomial measure on $\mathfrak{g}^\ast$. 
Example (1) $G$ a compact Lie group. Then
\[ H^*_c(pt) = \mathbb{C}[\Sigma, \ldots, f_n] \rightarrow H^*_c(M)^W \]
where $T$ is a maximal torus of $G$ and $W$ is the Weyl group of $G$.

Umkehr maps homomorphism: If $f: N \rightarrow M$ is a map of compact oriented manifolds then the pushforward
\[ f_*: H^*_c(N) \rightarrow H^*_{c+\dim M-\dim N}(M) \]
satisfies

(a) $(f \circ g)_* = f_* \circ g_*$
(b) $f_*(v f^*(u)) = (f_* v) u$
(c) If $f: N \rightarrow M$ is a fibration then $f_*$ corresponds to integration over the fibre.

The Thom isomorphism is
\[ H^*_c(M, M - N) \cong H^*_c(N) \]
where $\nu_N$ is the normal bundle to $N$ in $M$.

The Thom class is $\tilde{e}(N)$. The Euler class is $\tilde{f}_* \tilde{e_c}$.

Remark: If $f: N \rightarrow pt$ then $f_*: H^*_c(N) \rightarrow H^*_c(pt)$ is a homomorphism of multiplicative cohomology rings.
Localization \[ H^*_f(M) \to H^*_f(M) \]

If \( f = (x_1, \ldots, x_n) \) then \( H^*_f(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n] \).

And \( H^*_f(M) \) is a \( \mathbb{C}[x_1, \ldots, x_n] \)-module.

Localization is a functor:

\[ \{ \mathbb{C}[x_1, \ldots, x_n]\text{-modules} \} \to \{ \text{sheaves on } \mathbb{C}^n \} \]

\[ M \to \overline{M} \]

where the stalk at \( f \) of \( \overline{M} \) is \( \overline{M}_f = M \otimes_{\mathbb{C}[x_1, \ldots, x_n]} \mathbb{C}[x_1, \ldots, x_n] \).

The support of \( M \) is

\[ \text{supp}(M) = \bigcap_f V_f \text{ where } V_f = \{ v \in \mathbb{C}^n \mid f(v_1, \ldots, v_n) = 0 \} \]

where the intersection is over \( f \in \mathbb{C}[x_1, \ldots, x_n] \) s.t. \( f \neq 0 \).

Then \( \text{supp}(M) \subseteq \mathbb{C}^n \) where \( \mathbb{C}^n = \mathbb{C} \otimes_{\mathbb{R}} \text{Lie}(T) \).