The Glass Bead Game: A short talk for the vacation scholars at Univ. of Melbourne 18.12.2009

The simply laced Dynkin diagrams are

\[ \overset{D_{n-1}}{1 \cdots n-1} \]

\[ \overset{D_n}{1 \cdots n-2 \cdots n-1} \]

\[ \overset{E_6}{\cdots} \]

\[ \overset{E_7}{\cdots} \]

\[ \overset{E_8}{\cdots} \]

The glass bead game.

A sequence \((i_1, i_2, \ldots, i_k)\) represents a placement of \(k\) beads on runners \(i_1, i_2, \ldots, i_k\).

Fix an end configuration of beads \(\lambda\), the shape.

A standard tableau of shape \(\lambda\) is

\[ T = (i_1, i_2, \ldots, i_k) \]

such that the resulting bead configuration is \(\lambda\).
Example

The shape

\[ \lambda = \begin{array}{ccccccc}
5 & 4 & 3 & 5 & 5 & 4 & 4 \\
2 & 3 & 2 & 4 & 3 & 2 & 5 \\
1 & 1 & 1 & 1 & 1 & 2 & 3 \\
\end{array} \]

has 5 standard tableaux.

A skew shape is a shape such that any two beads on the same runner are separated by two beads, i.e.,

if

then

or

\[ \lambda \]

let \( \lambda \) be a skew shape.

Make a vector space

\[ \mathbb{R}^\lambda \] with basis \( \{ v_{\lambda} | \text{\lambda is a standard tableau of shape } \lambda \} \).
Define operators on $V$:

$$e_s V_T = \begin{cases} \sum V_T, & \text{if } s = T \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{i,j} V_T = \begin{cases} V_{sT}, & \text{if } sT \text{ is a standard tableau of shape } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

where

$$s_j (i_1, i_2, \ldots, i_l, j_l, \ldots, i_l = (i_1, i_2, \ldots, i_l, j_l, i_l, \ldots, i_l).$$

These operators are solutions to the Quiver Yang-Baxter equations:

Orient the edges of $D$. Then

$$\left( \psi_{j_l} \psi_{i_l} \psi_{i_l} - \psi_{j_l} \psi_{i_l} \psi_{i_l} \right) e_T = \begin{cases} e_T, & \text{if } j_{l+1} = i_j \text{ and } i_j \to i_{j+1} \\ -e_T, & \text{if } j_{l+1} = i_j \text{ and } i_{j+1} \to i_j \\ 0, & \text{otherwise} \end{cases}$$

if $T = (i_1, i_2, \ldots, i_l)$.
Remark The braid group has elements

and product $b_i b_j = \begin{array}{c} b_i \\ b_j \end{array}$

The braid group is generated by simple twists

$T_i : \begin{array}{c} 1 \ldots i \ldots n \\ \end{array}$

which satisfy

$T_{ji} T_{j} T_{ji} = T_{j} T_{ji} T_{j}$

the braid relation,
the Coxeter relation,
the Artin relation,
the Yang-Baxter equation

Let $b_1, \ldots, b_n$ be letters. Let

$b_T = b_{i_1} \cdots b_{i_k}$ for a standard tableau $T=(i_1, \ldots, i_k)$.

Then

$\sum_T b_T$, where the sum is over all standard tableaux of shape $\lambda$

is a canonical basis element in the quantum group