Let \( \mathbb{Z}^* \) be a lattice in \( \mathbb{R}^n \).

A set \( \Sigma \subseteq \mathbb{Z}^* \) is **convex** if it satisfies:

- if \( x, y \in \Sigma \) and \( x \in \mathbb{Z}^* \), then \( \lambda x + (1-\lambda)y \in \Sigma \)

Let \( \Sigma \subseteq \mathbb{Z}^* \). The **convex hull** of \( \Sigma \) is the subset \( \text{conv}(\Sigma) \) of \( \mathbb{R}^n \) such that:

- \( \text{conv}(\Sigma) \) is convex and \( \text{conv}(\Sigma) \supseteq \Sigma \),
- \( \text{conv}(\Sigma) \) is the smallest convex set containing \( \Sigma \).

An **integer polytope** \( P \) is the convex hull of a finite subset of \( \mathbb{Z}^* \).

The **normal fan** to \( P \) is

\[
\Delta_P = \{ \sigma^* \mid \sigma \text{ is a face of } P \}
\]

where

\[
\sigma^* = \{ \mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{x} \rangle \geq \langle \mathbf{v}, \mathbf{x}' \rangle \text{ for } \mathbf{x} \in \sigma, \mathbf{x}' \in P \}
\]

Let

\[
\mathcal{O}[\Delta_P \cap \mathbb{Z}^*] = \mathbb{C}\text{-span}\{ \mathbf{X}^\lambda \mid \lambda \in \sigma^* \cap \mathbb{Z}^* \}
\]

with \( \mathbf{X}^\lambda \mathbf{X}^{\lambda'} = \mathbf{X}^{\lambda + \lambda'} \) and let

\[
\mathcal{U}_{\sigma^*} = \text{Spec} \left( \mathcal{O}[\Delta_P \cap \mathbb{Z}^*] \right)
\]

The **tropical variety** of \( \Delta \) is

\[
X(\Delta) = \bigcup_{\sigma^*} \mathcal{U}_{\sigma^*} \text{ with } \mathcal{U}_{\sigma^*} \text{ and } \mathcal{U}_{\sigma^*} \text{ glued along } \mathcal{U}_{\sigma^*} \]
let \( v_1, \ldots, v_d \) be the rays of \( \Delta \)
\( x_1, \ldots, x_d \), \( x_i \) with \( x_i \) the first lattice point along \( v_i \).

\[ D_i = \overline{O_{x_i}} \]

and \( a_i \) be such that

\[ P = \{ u \in \mathbb{R}^d \mid \langle u, x_i \rangle + a_i \geq 0 \text{ for } 1 \leq i \leq d \} \]

Then

\[ D = a_1 D_1 + \cdots + a_d D_d \]

is a divisor on \( X(\Delta) \) that corresponds to a line bundle \( L \) on \( X(\Delta) \).

There is a bijection

\[ \{ \text{integer polytopes} \} \leftrightarrow \{ \text{pairs } (X, L) \text{ where } \}
\]

\[ X \text{ is a toric variety and } L \text{ is an ample line bundle on } X \]

Further let

\[ x_t \in \text{basis of } H^0(X, L^{\otimes t}) \]

Then

\[ \text{Card}(x_t) = \text{Card}(K P \cap \mathbb{Z}^d) \]
Examples of when global sections of line bundles is interesting

Flag varieties Internal Working seminar 02.06.2010 Melbourne Univ.

1. Complex reductive algebraic group
   \( G \)
2. Borel subgroup
   \( B \)
3. Maximal torus
   \( T \)

**Theorem** Let \( \lambda \) be a dominant weight of \( T \)

The irreducible finite dimensional \( \mathfrak{g} \)-modules are

\[ H^0(G/B, L_\lambda) \]

for dominant integral weights \( \lambda \).

where \( L_\lambda = G \times_B C_\lambda \) is a line bundle on \( G/B \)

and \( C_\lambda \) is the one-dimensional \( B \)-module coming from the character \( \chi^\lambda: T \to \mathbb{C}^* \) indexed by \( \lambda \).

Let \((W_\lambda, \Sigma_\lambda)\) be the \( \mathfrak{g} \)-reflection group

corresponding to \((G,T)\) and let \( C \) be the chamber of \( \Sigma_\lambda \) corresponding to \( B \)

Let \( \Sigma^\lambda, \ldots, \Sigma^\mu \) be the walls of \( C \).

A path on \( \Sigma_\lambda^* \) is a piecewise linear map \([0,1] \to \Sigma_\lambda^* \)

such that \( p(0) = \Sigma \) and \( p(1) \in \Sigma_\lambda^* \).
The root operators $\tilde{\tau}_1, \ldots, \tilde{\tau}_n$ are given by

$\tilde{\tau}_1, \ldots, \tilde{\tau}_n$.

Let $B(\Gamma)$ be the crystal generated by $\tau_1$

where $\tau_1(0) = 0$, $\tau_1(1) = 1$ and $\tau_1 \leq C$.

Then

$\text{char} \left( H^0(G/B, \mathcal{L}_\lambda) \right) = \sum_{\mu} \text{Corr}(B(\Gamma), \mathcal{L}_\mu) x^\mu$

$\mu \in \mathfrak{h}^*_C$.

Example, $G = SL_3(\mathbb{C})$
Let $LG^{v} = G^{v}(G^{v}(E))$.

$G^{v}(E) \cong \frac{\mathcal{G}^{v}(E)}{\mathcal{G}^{v}(E) \cdot E}$

$LG^{v}/K^{v}$ is the loop Grassmannian

$LG^{v}/K^{v}$ is the affine flag variety

Let $W = W_{0} \times \mathbb{Z}^{\mathbb{Z}}$ with $X^{v} = X^{v} \oplus \mathbb{Z}$.

Then

$LG^{v} = \bigcup \mathcal{K}^{v} \mathcal{K}^{v}$

$\mathcal{K}^{v} \cap U^{-1} \mathcal{K}^{v}$

$LG^{v} = \bigcup \text{Im}^{v} \text{Im}^{v}$

$\text{Im}^{v} \cap U^{-1} \text{Im}^{v}$

and the $MV$-intersections are

$K^{v} \cap K^{v} \cap U^{-1} \mathcal{K}^{v}$

and

$\text{Im}^{v} \cap U^{-1} \text{Im}^{v}$

An $MV$-cycle is an irreducible component of

$K^{v} \cap K^{v} \cap U^{-1} \mathcal{K}^{v}$ in $LG^{v}/K^{v}$. 
Then, let $\hat{G}_\mu$ be a basis of $\mathcal{H}(G/B, X^\mu)$. Then

$$\text{Card}(\hat{G}_\mu) = \text{Card} (B(\lambda)_\mu) = \text{Card} (\text{Inv} (K^\tau K \cap U^{-\tau_\mu} K))$$

By Cassens-Littelmann and we know explicitly the bijection

$$B(\lambda)_\mu \leftrightarrow \text{Inv} (K^\tau K \cap U^{-\tau_\mu} K)$$
Modular Forms

Let \( \Lambda \) be a lattice of rank \( 2g \) in \( \mathbb{C}^g \).

An abelian variety of dimension \( g \) is \( \mathbb{C}^g / \Lambda \) which can be embedded into projective space.

An elliptic curve is an abelian variety with \( g = 1 \).

A polarized abelian variety is a pair \( (T, L) \) where

- \( T \) is an abelian variety
- \( L \) is an ample line bundle on \( T \)

Theta functions are elements of \( H^0(T, L) \).

Let \( \mathfrak{g} = \mathfrak{sp}(2g, \mathbb{R}) \) with \( \mathfrak{g}^+ = \mathfrak{sp}(2g, \mathbb{R})^+ \) and \( A = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \).

There is a bijection

\[
\{ \text{polarized abelian varieties} \} \leftrightarrow \{ \text{polarized Hodge structures} \} \ 	ext{of weight } 1
\]

The Siegel upper half plane of degree \( g \) is

\[
G_g = \{ \tau \in \mathbb{H}^g \mid \tau^t = \tau \text{ and } \text{Im} \tau > 0 \}
\]

\( G_g \) is the period domain for polarized Hodge structures of weight \( 1 \).

Then \( G_g \sim \mathbb{H}^g \mathbb{H} \mathbb{H}^g \mathbb{H} \mathbb{H}^g \) and

\[
\mathbb{G}_g = \mathbb{Sp}(2g, \mathbb{R}) / \mathbb{K} \text{ where } \mathbb{K} = \mathbb{Q}^{??} \text{ (a complete)}
\]
Let $d_1, \ldots, d_g \in \mathbb{Z}_{>0}$ with $d_1 + \cdots + d_g$ and $A = \begin{pmatrix} d_1 & \cdots & 0 \\ 0 & \cdots & d_g \end{pmatrix}$.

$\text{Sp}(A, \mathbb{Z}) = \{ M \in \text{GL}(2g, \mathbb{Z}) | M (0 A)^T M^T = (0 A) \}$

$\Gamma_A(n) = \{ M \in \text{Sp}(A, \mathbb{Z}) | M \equiv I_{2g} \pmod{n} \}$

Then

$\text{Sp}(A, \mathbb{Z}) \backslash G_g = \{ \text{polarized abelian varieties} \} = A_g$

$\Gamma_A(n) \backslash G_g = \{ \text{level } n \text{ polarized abelian varieties of type } d_1, \ldots, d_g \} = A_g(n)$

Modular forms of level $n$ and weight $k$ are elements of $H^0(Z_g(n), \mathbb{C}(k))$. 