Equivalence of categories

\[ \text{Chevalley} \] \[ \text{groups } G \]
\[ \text{Z-reflection?} \] \[ \text{groups } W_0, \frac{Z}{2}, \frac{Z}{2}, \frac{Z}{2} \]

\[ G \text{ is generated by } x_v(), x_w(), h_{uvld} \]
\[ B \text{ is generated by } x_v(), h_{uvld} \]
\[ T \text{ is generated by } h_{uvld} \]

and our favourite example is:

\[ GL_2(\mathbb{F}) \equiv \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \equiv \left\{ \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix} \right\} \]

We chose \( C \) so that

\( W_0 \) is presented by \( s_1, \ldots, s_n \) with

\[ s_i^2 = 1 \text{ and } s_i s_{ij} \ldots s_{ij} = s_{ij} \ldots s_{ij} \]

Example:

\[ W_0 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 
\end{array} \]

\[ \begin{array}{cccc}
\ell & c & d & e \\
\ell & \ell & \ell & \ell \\
\ell & \ell & \ell & \ell \\
\ell & \ell & \ell & \ell 
\end{array} \]

\[ R^+ = \mathbb{F} x, x_v, g \]

\[ K = s_v s_i \text{ is an element of } W_0 \]

\[ z_f = \frac{x}{y}(5) x_v^{-1} \frac{x}{y} \text{ and } B \text{ is a point on } G/B. \]
$G/B$ is the flag variety

$G = \bigcup_{w \in W} B_w B$

and if $\bar{w}_i = x_{i-1}(1) x_i(1) x_{i+1}(1)$ and $w = s_i \cdots s_{i+k}$ is a minimal length path to $w$ then

$\text{Fix}_B(\bar{w}_i) \cdot x_{i-1}(1) \cdot \cdots \cdot x_i(1) \cdot \bar{w}_i^{-1} B | g, \ldots, g \in F^2$

are the points of $B_w B \subset F^2$ in $G/B$.

$B_w B$ are the Schubert cells

$\overline{B_w B}$ are the Schubert varieties

$n_w B = x_{i-1}(1) \bar{w}_i^{-1} \cdots x_i(1) n_{i+1}^{-1} B$

is the unique $T$-fixed point in $B_w B$ (the "center of the cell").

$\overline{B_w B} = \bigcup_{v \leq w} B_v B$ (Bruhat order).
A generalised cohomology theory is a family of functors

\[ E_T : \text{T-spaces} \rightarrow \mathcal{C} \]

indexed by groups \( T \), which satisfy

**Axiom (Künneth = products)** something like

\[ E_{\mathcal{G} \times \mathcal{H}} (X \times Y) \rightarrow E_{\mathcal{G}} (X) \otimes E_{\mathcal{H}} (Y) \]

**Axiom (Change of groups)** something like

\[ E_{\mathcal{G}} (X \vee Y) \rightarrow E_{\mathcal{H}} (X) \]

**Axiom (Suspension = periodicity)** something like

\[ E_{\mathcal{G}} (S^n \times X) \rightarrow E_{\mathcal{G}} (S^n) \times E_{\mathcal{G}} (X) \].
K-theory = cohomology

Let $T$ be a group and $X$ a space.

$K_T(X)$ is Grothendieck group of $T$-equivariant vector bundles on $X$.

i.e. generators $[V]$ with relations

$[V] = [W]$ if $V = W$,

$[V] + [W] = [V \oplus W]$, $[V][W] = [V \otimes W]$,

$[U] - [V] + [W] = 0$ if $0 \to U \to V \to W \to 0$ is exact.

Then

(a) $K_T(\mathbb{A}) = R(T)$, the Grothendieck ring of $T$-modules.

(b) $K_T(X)$ is a $K_T(\mathbb{A})$-module.

(c) $K_T(G/B)$ has $R(T)$-basis $\{[U_v] \mid v \in W_0\}$

where $U_v$ is the structure sheaf of $X_v = \overline{BvB}$.

(d) The Chern character is an isomorphism

$K_T(G/B) \xrightarrow{\text{ch}} H_T^*(G/B)$

$[L] \mapsto 1 + c_1(L) + \frac{1}{2} c_1(L)^2 + \frac{1}{3!} c_1(L)^3 + \cdots$

for line bundles $L$ and $c_i(L) =$ Chern class of $L$. 
Remarks

(a) A $T$-equiv vector bundle $V$ is a $T$-module.

$T$-modules $V$ have composition series.

The simple $T$-modules are $C_\mu$ correspond to

$$X^\mu: T \to C^x$$

$$h_\mu(x) \mapsto \mu(x)$$

for $\mu \in \text{Hom}_d (\mathbb{Z}_d, \mathbb{C})$

where $\mu(x)$ means $\mu(x^d)$. So

$$R(T) = K_f(pt) = \text{span}\{X^\mu | \mu \in \mathbb{Z}_d^*\} \text{ with } X^\mu X^\nu = X^{\mu+\nu}.$$

(b) By Künneth's $pt \times X \cong X$ gives

$$K_f(pt) \otimes K_f(X) \to K_f(X).$$

(c) See Grothendieck 1958 Prop 7;

$U_{W_v}$ comes from $X_v \to pt$ and $X_v \to G/B$

and $G = U_{B_n} B$ (affine paving)

causes $\{U_{W_v} | v \in W_0\}$ to be a basis of $K_f(G/B)$

(d) What is cohomology.
The nil affine Hecke algebra

Let $\mathfrak{g}^* = \text{Hom}_{\mathbb{Z}}(\mathfrak{g}, \mathbb{Z})$. The affine Weyl group is

$$W = \{ x^{\mu} t_w / \mu \in \mathfrak{g}^*, w \in W_0 \}$$

with

$$x^{\mu} x^{\nu} = x^{\mu + \nu}, \quad t_w t_v = t_{wv}, \quad t_w x^{\mu} = x^{w \mu} t_w.$$

Let

$$R(T) = \text{span} \{ x^{\mu} t_w / \mu \in \mathfrak{g}^*, w \in W_0 \},$$

$$Q(T) = \text{field of fractions of } R(T),$$

$$K = \text{span} \{ x^{\mu} t_w / \mu \in \mathfrak{g}^*, w \in W_0 \} = R(T) \text{span} \{ t_w / w \in W_0 \},$$

$$K^* = Q(T) - \text{span} \{ t_w / w \in W_0 \}$$

where so that $K$ is the group algebra of $W$. If

$$\Delta_i = \frac{1}{1-x^{-1} - t_i} \quad \text{and} \quad \Delta_i^\vee = \frac{1}{1-x^{t_i} - t_i},$$

then

$$\Delta_i^2 = \Delta_i \quad \text{and} \quad \Delta_i^\vee = -\Delta_i.$$

Because of these relations $K$ or $K^*$ is often called the nil affine Hecke algebra.
"Combinatorial" realization of $K_T(\mathbb{C}/B)$

Define $\Psi \in \text{Fun}(W_0, R(T))$ by

$$t_w = \sum_{v \in W_0} \Psi(v) \Delta_w$$

and set

$$\Psi = R(T)\text{-span} \{ \Psi(v) | v \in W_0 \}.$$

**Theorem (a) GKH condition:**

$$\Psi = \{ \Psi \in \text{Fun}(W_0, R(T)) \mid \Psi(s_w) - \Psi(w) \in W^{-1} X^\vee R(T) \}$$

for $x \in \mathbb{R}^+$, $w \in W_0$.

where $s_w$ is the reflection in $W_0$ correspond to $x \in \mathbb{R}^+$.

(b) With pointwise product on $\text{Fun}(W_0, \mathbb{R}(T))$:

$$K_T(\mathbb{C}/B) \rightarrow \Psi$$

$$[\mathcal{C}_X] \mapsto \Psi \text{ is an } R(T)\text{-algebra isomorphism.}$$

(c) With $\omega: pt \rightarrow \mathbb{C}/B$ so $\omega^*: K_T(\mathbb{C}/B) \rightarrow K_T(pt)$

then

$$\Psi(w) = \omega^* \left( [\mathcal{C}_X] \right).$$
Example $SL_3(\mathbb{C}) = G$

Since $W_0 = \text{hexagon}$, $g \in \text{Fin}(W_0, R/I(T))$

is a hexagon with chamber $w$ labeled $g(w)$

Then $K_g(\mathbb{C}/B)$ has basis

$[L_{x_1}] = 1-x^y \quad 1-x^y$

$[L_{x_2}] = (1-x^y)^2(1-x^y)$

$[L_{x_3}] = (1-x^y)^2(1-x^y)$

$[L_{x_4}] = (1-x^y)^2(1-x^y)$

$[L_{x_5}] = (1-x^y)^2(1-x^y)$

$[L_{x_6}] = (1-x^y)^2(1-x^y)$

$[L_{x_7}] = (1-x^y)^2(1-x^y)$