In progress with A. Ghitza and S. Kannan.

Compare $H^0(\mathcal{E}_b, \mathcal{L}_b)$ and MV cycles of type $1$.


$Lustig$,
$Khoroshkin-Leclerc-Renquier$

Lusztig,
Kashwara-Seito
Geiss-Leclerc-Schroer

Quiver Hecke algebra
modules $\mathcal{L}_b$

Reprojective algebra
modules $\mathcal{L}_b$

MV cycles $\mathcal{E}_b$
The shuffle algebra $\mathcal{C}N^1$.

Let $F$ be the free algebra generated by $t_1, \ldots, t_n$. The shuffle product $\circ : F \times F \to F$ is given by

$$u \circ v = \sum_{\sigma \in S_{|u|} \times S_{|v|}} \sigma(u \cdot v),$$

for words $u = t_{i_1} \cdots t_{i_m}$ and $v = t_{j_1} \cdots t_{j_n}$, where the sum is over minimal length coset representatives for cosets in $S_{|u|} \times S_{|v|}$. For example

$$f(t_1 t_2) = t_1 t_2 t_1 + t_1 t_2 t_2 + t_2 t_1 t_1 + t_2 t_1 t_2.$$

$\mathcal{C}N^1$ is the subalgebra of $F$ generated by $t_1, \ldots, t_n$.

$t_1, \ldots, t_n$ correspond to simple coroots $\alpha_1, \ldots, \alpha_n$ in a root system.

Favourite root system: $\alpha_1 = \alpha_2 = \zeta = -e_1 + 1$, $\langle \alpha_1, \alpha_2 \rangle = 2$.

$W_0 = S_n$, $s_i = \begin{pmatrix} 1 & 1 & 0 \ldots \ 0 \end{pmatrix}$, $w_0 = \begin{pmatrix} 0 & 1 \ 1 & 0 \end{pmatrix}$.

The MV-polytope corresponding to $\text{ch}(L_b)$ is the convex hull of the paths/terms in $\text{ch}(L_b)$.

Example: $t_1 t_2$ and $\text{ch}(L_b) = \text{fix}_0 h_1$ has MV-polytope $d = \mathcal{C}N^1$. 
Quiver Hecke algebra modules

The Khovanov-Lauda-Rouquier, or quiver Hecke, algebra $R_d$ has generators

$$e_u, y_1, \ldots, y_d, \psi_1, \ldots, \psi_d,$$

where $u = t_1 \cdots t_d$ runs over words of length $d$

$$e_u e_v = \delta_{uv} e_u \quad \text{and} \quad \sum u e_u = 1$$

$y_1, \ldots, y_d$ are like Murphy elements

$\psi_1, \ldots, \psi_d$ are like simple transpositions in $S_d$.

$R_d$ is $\mathbb{Z}$-graded! For a $\mathbb{Z}$-graded $R_d$-module $M$,

$$M = \bigoplus_{i} M[i] = \bigoplus_{i} \bigoplus_{u} e_u M[i]$$

and the character of $M$ is

$$\text{ch}(M) = \sum_{i} \sum_{u} \dim(e_u M[i]) q^{i u} \quad \text{in the q-shuffle algebra.}$$

Kleshchev-Ram: The simple homogeneous $R_d$ module corresponding to

$b = \begin{array}{ccccccc}
5 & 4 & 4 & 5 & 5 & 4 & 4 \\
2 & 3 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}$

has dimension the number of standard tableaux of shape $d$

$$\text{ch}(b) = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9$$
An MV polytope is

\[ b = \text{convex hull of } \mu w \mid w \in W_0 \text{ of its vertices}. \]

For a minimal length path \( W_0 = s_i \cdots s_N \to W_0 \)
the i-perimeter or Lustig parametrization of \( b \) is

\[ \text{per}_i (b) = (b_i, b_i, \ldots, b_i) \]

the sequence of lengths \( \mu, \frac{1}{2} \mu_i, \mu_i, \frac{1}{2} \mu_i, \mu_i \to \ldots \)

Any per_i (b) can be computed from per_i (b) by
a sequence of "Chevalley relations":

\[ R_{ii} = (b_i, b_i, b_i) \]
\[ R_{ij} (b_i, b_i, b_i) = (b_i + b_i - b_i, b_i, b_i) \]

(see Morier-Genoud Thesis).

The crystal operator \( \tilde{t}_i \) is given by

\[ \text{per}_i (\tilde{t}_i b) = (b_i + 1, b_i, \ldots, b_i) \]

and the i-growth, or string parametrization of \( b \) is

\[ d = \tilde{t}_i^N \cdots \tilde{t}_i^N b_i, \text{ where } d_i = 0. \]
The MV cycles

\[ C(I(l)) = \{ a_{-l}t^l + a_{l-1}t^{l-1} + \cdots + a_i t^i \} \quad \text{for } a_i \in \mathbb{C}, \quad -l \leq i \leq l \]

\[ C(I(l)) = \{ a_0 + a_1t + a_2t^2 + \cdots \} \quad \text{for } a_i \in \mathbb{C} \]

\[ G = \text{GL}_{n+1}(\mathbb{C}(l)) \quad \text{and} \quad K = \text{GL}_n(I(l)) \]

\[ U = \left\{ \begin{pmatrix} I & \cdot \\ \cdot & \cdot \end{pmatrix} \right\} \subseteq \text{GL}_{n+1}(\mathbb{C}(l)) \]

Let

\[ t_{\lambda \mu} = \begin{pmatrix} t_{\lambda 1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{\lambda n} \end{pmatrix} \quad y_{i|\lambda \mu}(t) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{i|\lambda \mu} \end{pmatrix} \]

for \( \lambda, \cdots, \lambda \in \mathbb{Z} \) and \( a_i \in \mathbb{C}, \quad j \in \mathbb{Z} \).

\( G/K \) is the loop Grassmanian.

The Cartan and Iwasawa decompositions are

\[ G = \bigcup_{\lambda} K t_{\lambda \mu} K \quad \text{and} \quad G = \bigcup_{\mu} U^{-1} t_{\mu \nu} K \]

The MV cycles of type \( \lambda \mu \) and weight \( \mu \) are the irreducible components

\[ Z_{\mu} \in \text{Inv} \left( K t_{\lambda \mu} K \cap U t_{\mu \nu} K \right) \]
Composition series: \( ch(Z_b) \)

The MV cycles are indexed by MV polytopes and by Barman-Caussenet,

if \( b = \cdots \hat{b_i} \cdots \hat{b_j} \) then

\[
Z_b = y_{i_1} (t^{\alpha_{i_1}} \text{circ} t^{\alpha_{i_1}'} \cdots y_{i_d} (t^{\alpha_{i_d}} \text{circ} t^{\alpha_{i_d}'} )K,
\]

where

\[
e_{i,j} = \langle a_{i,j}^{-1}, a_{i,j+1}, \ldots, a_{i,d}\rangle
\]

and

\[
\text{circ} t^{\alpha_i} = \{ a, a \cdot t^{-1} \cdot a_t \cdot t^t \mid a \in \mathbb{C}, a \cdot t^t \neq 0 \}
\]

let \( Z_b \) be an MV cycle of dimension \( d \).

A composition series for \( Z_b \) is

\[
\langle i_1, \ldots, i_1, i_d, \ldots, i_d \rangle
\]

such that

\[
Z_b = \left\{ y_{i_1} (a_t t^{i_1}) \cdots y_{i_d} (a_t t^{i_d}) K \mid a_t \in \mathbb{C} \right\}
\]

The character of \( Z_b \) is

\[
ch(Z_b) = \sum_{(i_1, \ldots, i_d)} t_{i_1} \cdots t_{i_d} \quad \text{an element of } \mathbb{C}[N].
\]