Math 127 A 2

Winter quarter, 2005

Notes on norms and inner product spaces

We quickly summarise some basic material about vector spaces. A vector space is a pair consisting of a field F of scalars (usually $\mathbb{R}$, but could be $\mathbb{C}$ or other types of fields) and a set of vectors V with operations of addition and scalar multiplication satisfying;

- $a + b = b + a$ for all $a, b \in V$
- $a + (b + c) = (a + b) + c$ for all $a, b, c \in V$
- there is a vector 0 satisfying $0 + a = a + 0 = a$ for all $a \in V$
- for every vector a there is a vector $-a$ satisfying $a+(-a) = (-a)+a = 0$
- for every $\lambda \in F$ and vectors $a, b \in V$, $\lambda(a + b) = \lambda a + \lambda b$
- for every $\lambda, \gamma \in F$ and vector $a \in V$, $(\lambda\gamma)a = \lambda(\gamma a)$
- for every vector $a \in V$, $1a = a$, where $1 \in F$ is the unit element.

A key fact about vector spaces is that every vector space has a *basis* i.e a set of vectors $\mathbb{B}$ satisfying;

every vector $a \in V$ can be written uniquely as a linear combination of vectors in $\mathbb{B}$, i.e $a = \sum \lambda_i b_i$, where $b_i \in \mathbb{B}$, $\lambda_i \in F$, for $1 \leq i \leq n$.

Vector spaces have many different bases $\mathbb{B}$ but all bases have the same cardinality, i.e number of vectors. If this number is finite, we say V is a finite dimensional vector space and otherwise that V is infinite dimensional.

Standard example

$\mathbb{R}^n$ has standard basis $\mathbb{B} = \{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)\}$. Every vector $a \in \mathbb{R}^n$ has the form $a = (a_1, a_2, \ldots a_n) = a_1 e_1 + \cdots + a_n e_n$.

Infinite dimensional example

Define l_2 to be the vector space of infinite real sequences $\{s_1, s_2, \ldots s_k, \ldots\}$ which are *square summable*. i.e $\sum {s_k}^2 < \infty$ is a convergent series. So for example, $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is an element of l_2 but $\{1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \dots\}$ is not. We can add square summable sequences and multiply them by real numbers so this is indeed a vector space. It has a huge basis! Note that taking $\{(1, 0, \dots, \}, \{0, 1, 0, \dots, \}, \dots,$ does *not* give a basis for l_2 since only sequences with finitely many non-zero terms are linear combinations of these sequences. So lots more sequences are needed to form a basis.

An inner product space is a real vector space V together with a product $<, >$ of pairs of vectors giving a real number, satisfying;

- $< a, b >=< b, a >$ for all $a, b \in V$
- $< a, b + c >=< a, b > + < a.c >$ for all $a, b, c \in V$
- $\lambda < a, b >=< \lambda a, b >$ for all $\lambda \in \mathbb{R}$, $a, b \in V$
- $< a, a >\geq 0$ with $< a, a >= 0$ if and only if $a = 0$, for all $a \in V$

Given an inner product, we then define a norm on V by $\|a\| = \sqrt{<a, a >}$
We can also define a normed space as a real vector space V together with a map $\|\|$ from V to $\mathbb{R}$ satisfying;

- $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in V$
- $|\lambda|\|a\| = \|\lambda a\|$ for all $\lambda \in \mathbb{R}$, $a \in V$
- $\|a\| \geq 0$ with $\|a\| = 0$ if and only if $a = 0$, for all $a \in V$

Standard examples

For $\mathbb{R}^n$, the standard inner product is $< a.b >= a_1b_1 + \dots a_nb_n$ i.e the dot product. The standard norm is then $\|a\| = \sqrt{({a_1}^2 + \cdots + {a_n}^2)}$. There are also norms which do not come from any inner product - for example on $\mathbb{R}^n$ we can also define

$$\|a\|' = \max\{|a_i| : 1 \leq i \leq n\}$$

It is pretty easy to show that this is a norm but it is harder to prove that there is no inner product producing it.

A very beautiful example is on l_2 as above. We can define $< s, s' >= \sum s_i {s'}_i$ Then it turns out this is an inner product (showing it is always a convergent series is done via the Cauchy Schwartz inequality!). Also the corresponding norm is $\|s\| = sqrt \sum {s_i}^2$. Note here, we are given the norm by definition of the sequences in l_2 and can use it to define the inner product.

Some exercises to think about norms and inner products

1. Define a product on $\mathbb{R}^3$ by $< a, b >= a_1b_1 - a_2b_2 + 3a_3b_3$. Show that this is not an inner product.

2. Define a product on $\mathbb{R}^3$ by $< a, b >= a_1b_1 + 2a_2b_2 + 3a_3b_3$. Show that this is an inner product.

3 Try to define $\|a\| = \sqrt{(}a_1{}^2 - a_2{}^2 + 3a_3{}^2)$ on $\mathbb{R}^3$. Show that this actually is not well-defined and is certainly not a norm.

4. Try to define $\|a\| = \sqrt{(}a_1{}^2 + 2a_2{}^2 + 3a_3{}^2)$ on $\mathbb{R}^3$. Show that this is well-defined and is a norm.

The Cauchy Schwartz inequality is fundamental - see if you can use it to justify the definition of the inner product on l_2. It says that for any inner product space, $| < a, b > | \leq \|a\|\|b\|$.