Math 127 A 2
Winter quarter, 2005

Notes on norms and inner product spaces

We quickly summarise some basic material about vector spaces. A vector space is a pair consisting of a field \( F \) of scalars (usually \( \mathbb{R} \), but could be \( \mathbb{C} \) or other types of fields) and a set of vectors \( V \) with operations of addition and scalar multiplication satisfying:

- \( a + b = b + a \) for all \( a, b \in V \)
- \( a + (b + c) = (a + b) + c \) for all \( a, b, c \in V \)
- there is a vector 0 satisfying \( 0 + a = a + 0 = a \) for all \( a \in V \)
- for every vector \( a \) there is a vector \(-a\) satisfying \( a + (-a) = (-a) + a = 0 \)
- for every \( \lambda \in F \) and vectors \( a, b \in V \), \( \lambda(a + b) = \lambda a + \lambda b \)
- for every \( \lambda, \gamma \in F \) and vector \( a \in V \), \( (\lambda \gamma) a = \lambda (\gamma a) \)
- for every vector \( a \in V \), \( 1 a = a \), where \( 1 \in F \) is the unit element.

A key fact about vector spaces is that every vector space has a basis i.e a set of vectors \( B \) satisfying:

- every vector \( a \in V \) can be written uniquely as a linear combination of vectors in \( B \), i.e \( a = \sum \lambda_i b_i \), where \( b_i \in B \), \( \lambda_i \in F \), for \( 1 \leq i \leq n \).

Vector spaces have many different bases \( B \) but all bases have the same cardinality, i.e number of vectors. If this number is finite, we say \( V \) is a finite dimensional vector space and otherwise that \( V \) is infinite dimensional.

**Standard example**

\( \mathbb{R}^n \) has standard basis \( B = \{ e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1) \} \). Every vector \( a \in \mathbb{R}^n \) has the form \( a = (a_1, a_2, \ldots a_n) = a_1 e_1 + \cdots + a_n e_n \).

**Infinite dimensional example**

Define \( l_2 \) to be the vector space of infinite real sequences \( \{ s_1, s_2, s_3, \ldots \} \) which are square summable, i.e \( \sum s_k^2 < \infty \) is a convergent series. So for
example, \{1, \frac{1}{2}, \frac{1}{3}, \ldots \} is an element of \(l_2\) but \(\{1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \ldots \}\) is not. We can add square summable sequences and multiply them by real numbers so this is indeed a vector space. It has a huge basis! Note that taking \(\{(1,0,\ldots),\{0,1,0,\ldots,\},\ldots\}\), does not give a basis for \(l_2\) since only sequences with finitely many non-zero terms are linear combinations of these sequences. So lots more sequences are needed to form a basis.

An inner product space is a real vector space \(V\) together with a product \(<, >\) of pairs of vectors giving a real number, satisfying:

- \(<a,b> = <b,a>\) for all \(a,b \in V\)
- \(<a,b+c> = <a,b> + <a,c>\) for all \(a,b,c \in V\)
- \(\lambda <a,b> = <\lambda a,b>\) for all \(\lambda \in \mathbb{R}, a,b \in V\)
- \(<a,a> \geq 0\) with \(<a,a> = 0\) if and only if \(a = 0\), for all \(a \in V\)

Given an inner product, we then define a norm on \(V\) by \(\|a\| = \sqrt{<a,a>}\). We can also define a normed space as a real vector space \(V\) together with a map \(||||\) from \(V\) to \(\mathbb{R}\) satisfying:

- \(||a + b|| \leq ||a|| + ||b||\) for all \(a,b \in V\)
- \(||\lambda||a|| = ||\lambda a||\) for all \(\lambda \in \mathbb{R}, a \in V\)
- \(||a|| \geq 0\) with \(||a|| = 0\) if and only if \(a = 0\), for all \(a \in V\)

**Standard examples**

For \(\mathbb{R}^n\), the standard inner product is \(<a,b> = a_1b_1 + \ldots a_nb_n\) i.e the dot product. The standard norm is then \(||a|| = \sqrt{a_1^2 + \cdots + a_n^2}\). There are also norms which do not come from any inner product - for example on \(\mathbb{R}^n\) we can also define

\[ ||a||' = \max\{|a_i| : 1 \leq i \leq n\} \]

It is pretty easy to show that this is a norm but it is harder to prove that there is no inner product producing it.

A very beautiful example is on \(l_2\) as above. We can define \(<s,s'> = \sum s_is'_i\). Then it turns out this is an inner product ( showing it is always a convergent series is done via the Cauchy Schwartz inequality!). Also the corresponding norm is \(||s|| = \sqrt{\sum s_i^2}\). Note here, we are given the norm by definition of the sequences in \(l_2\) and can use it to define the inner product.

**Some exercises to think about norms and inner products**

1. Define a product on \(\mathbb{R}^3\) by \(<a,b> = a_1b_1 - a_2b_2 + 3a_3b_3\). Show that this is not an inner product.
2. Define a product on $\mathbb{R}^3$ by $<a, b> = a_1b_1 + 2a_2b_2 + 3a_3b_3$. Show that this is an inner product.

3 Try to define $\|a\| = \sqrt{(a_1^2 - a_2^2 + 3a_3^2)}$ on $\mathbb{R}^3$. Show that this actually is not well-defined and is certainly not a norm.

4. Try to define $\|a\| = \sqrt{(a_1^2 + 2a_2^2 + 3a_3^2)}$ on $\mathbb{R}^3$. Show that this is well-defined and is a norm.

The Cauchy Schwartz inequality is fundamental - see if you can use it to justify the definition of the inner product on $l_2$. It says that for any inner product space, $|<a, b>| \leq \|a\|\|b\|$. 
