**Topic 3: INTEGRAL CALCULUS**

In this topic we learn several techniques of integration, and then consider some applications.

3.1 Standard Integrals

3.2 Integration using Trigonometric Identities

3.3 Integration using Partial Fractions



## 3.1 Standard Integrals

You should be familiar with the following integrals, where $k \in \mathbb{R}$:

Basic:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for } n \neq -1$$

("add one to the power and divide by what you get")

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

Trig:

$$\int \sin kx \, dx = -\frac{1}{k} \cos kx + C$$

$$\int \cos kx \, dx = \frac{1}{k} \sin kx + C$$

$$\int \sec^2 kx \, dx = \frac{1}{k} \tan kx + C$$



And since we learnt how to differentiate inverse trigonometric functions in Week 5, we also know:

Inverse Trig:

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

where $a > 0$.

Homework: Check these by differentiating the right hand side using the chain rule.



## 3.2 Integration using Trigonometric Identities [Chapter 8.3]

An integral of the form

$$\int \sin^m(x) \cos^n(x) \, dx$$

(where $m$, $n$ are nonnegative integers) can be solved using trigonometric identities. There are two cases to consider, depending on whether the powers $m$ and $n$ are even or odd.

### 3.2.1 Case 1: At least one of *m, n* is odd.

In this case we can turn the integral into a derivative present type. To do this, first split off a factor of the function with the odd power. This will become the derivative, so we express everything else in terms of the *other* function, by using the identities

$$\sin^2(x) = 1 - \cos^2(x) \qquad \text{and} \qquad \cos^2(x) = 1 - \sin^2(x) .$$

It is best illustrated by example!

**Example:** $\int \sin^2(x)\cos^5(x)\,\mathrm{d}x$

We split off a factor of the function with the odd power, $\cos(x)$, then express the other powers of cos in terms of sin by using the identity $\cos^2(x) = 1 - \sin^2(x)$.

$$= \int \sin^2 x \cos^4 x \cos x \, dx$$

$$= \int \sin^2 x \left(\cos^2 x\right)^2 \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

This is then a derivative present type integral which can be solved by setting $u = \sin(x)$...



let $u = \sin x \quad \Rightarrow \quad \frac{du}{dx} = \cos x$

"$du = \cos x \, dx$"

$$= \int u^2 (1-u^2)^2 \, du$$

$$= \int u^2 (1 - 2u^2 + u^4) \, du$$

$$= \int u^2 - 2u^4 + u^6 \, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$



**Example:** $\int \sin^7(2x)\,\mathrm{d}x$

$$= \int \sin^6(2x) \sin(2x) \, dx$$

$$= \int \left(\sin^2 2x\right)^3 \sin 2x \, dx$$

$$= \int \left(1 - \cos^2 2x\right)^3 \sin 2x \, dx$$

Let $u = \cos 2x \quad \Rightarrow \quad \frac{du}{dx} = -2\sin 2x$

"$du = -2\sin 2x \, dx$"

"$-\frac{1}{2}du = \sin 2x \, dx$"



$$= \int (1-u^2)^3 \left(-\frac{1}{2}du\right)$$

$$= -\frac{1}{2}\int 1 - 3u^2 + 3u^4 - u^6 \, du$$

$$= -\frac{1}{2}\left(u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7\right) + C$$

$$= -\frac{1}{2}u + \frac{1}{2}u^3 - \frac{3}{10}u^5 + \frac{1}{14}u^7 + C$$

$$= -\frac{1}{2}\cos 2x + \frac{1}{2}\cos^3 2x - \frac{3}{10}\cos^5 2x + \frac{1}{14}\cos^7 2x + C$$

### 3.2.2 Case 2: Both *m* and *n* are even.

For this case, we need to make use of some different trigonometric identities.

Recall from the double angle formula for cos, we have

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= (1 - \sin^2(x)) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \\ \Rightarrow \qquad 2\sin^2(x) &= 1 - \cos(2x) \end{aligned}$$

Hence

$$\sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big)$$

**Homework**: Derive the corresponding identity for $\cos^2(x)$:

$$\cos^2(x) = \frac{1}{2}\big(1 + \cos(2x)\big)$$



So, for integrals of the form $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$ where $m$ and $n$ are both even, we make use of the trigonometric identities:

$$\boxed{\begin{aligned} \sin^2(x) &= \frac{1}{2}\big(1 - \cos(2x)\big) \\ \cos^2(x) &= \frac{1}{2}\big(1 + \cos(2x)\big) \end{aligned}}$$

until we obtain standard trigonometric integrals.



**Example:** $\int \cos^4(x)\,\mathrm{d}x$

(Note that $\sin x$ has power 0, so both powers are even.)

$$\begin{aligned} &= \int (\cos^2 x)^2\, dx \\ &= \int \left(\tfrac{1}{2}(1 + \cos 2x)\right)^2 dx \\ &= \int \tfrac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) dx \\ &= \tfrac{1}{4}\int 1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\, dx \\ &= \tfrac{1}{4}\int \tfrac{3}{2} + 2\cos 2x + \tfrac{1}{2}\cos 4x\, dx \\ &= \tfrac{1}{4}\left(\tfrac{3}{2}x + \sin 2x + \tfrac{1}{8}\sin 4x\right) + C \\ &= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \end{aligned}$$



**Example:** $\int \sin^2(3x)\cos^2(3x)\,\mathrm{d}x$

$$\begin{aligned} &= \int \tfrac{1}{2}(1 - \cos 6x)\cdot\tfrac{1}{2}(1 + \cos 6x)\, dx \\ &= \tfrac{1}{4}\int 1 - \cos^2 6x\, dx \\ &= \tfrac{1}{4}\int \sin^2 6x\, dx \\ &= \tfrac{1}{4}\int \tfrac{1}{2}(1 - \cos 12x)\, dx \\ &= \tfrac{1}{8}\int 1 - \cos 12x\, dx \\ &= \tfrac{1}{8}\left(x - \tfrac{1}{12}\sin 12x\right) + C \\ &= \tfrac{1}{8}x - \tfrac{1}{96}\sin 12x + C \end{aligned}$$

**Homework:** Solve the following:

(a) $\int \sin^2(x)\,dx$ (b) $\int \sin^3(x)\,dx$ (c) $\int \sin^4(x)\,dx$

Answers: (a) $\frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

(b) $\frac{1}{3}\cos^3(x) - \cos(x) + C$

(c) $\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$



### 3.2.3 Powers of tan and sec

*Similar* methods can often be applied to integrals involving tan and sec, by using the identity

$$\boxed{\tan^2(x) + 1 = \sec^2(x)}$$

and recalling that $\sec^2(x)$ is the derivative of $\tan(x)$.

**Example:** $\int \tan^2(x)\,dx$

$= \int \sec^2 x - 1\,dx$

$= \tan x - x + C$



**Example:** $\int \tan^3(3x)\sec^4(3x)\,dx$

$= \int \tan^3 3x \cdot \sec^2 3x \cdot \sec^2 3x\,dx$

$= \int \tan^3 3x\,(\tan^2 3x + 1)\sec^2 3x\,dx$

Let $u = \tan 3x$ $\quad \frac{du}{dx} = 3\sec^2 3x$

" $du = 3\sec^2 3x\,dx$ "

" $\frac{1}{3}du = \sec^2 3x\,dx$ "

$= \int u^3(u^2+1)\left(\frac{1}{3}du\right)$

$= \frac{1}{3}\int u^5 + u^3\,du$

$= \frac{1}{3}\left(\frac{1}{6}u^6 + \frac{1}{4}u^4\right) + C$

$= \frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + C$



**Exercise:** $\int \sec^6(2x)\,dx$

Answer: $\frac{1}{10}\tan^5(2x) + \frac{1}{3}\tan^3(2x) + \frac{1}{2}\tan(2x) + C$

**Homework:** $\int \sec^4(5x)\tan(5x)\,dx$

Answer: $\frac{1}{20}\tan^4(5x) + \frac{1}{10}\tan^2(5x) + C$

### Additional questions

You can now attempt a selection of exercises 1 - 12 and 29 - 42 from Chapter 8.3 in the textbook.



### 3.3 Integration using Partial Fractions [Chapter 8.5]

The method of **partial fractions** can be used to solve integrals of the form

$$\int \frac{f(x)}{ax^2+bx+c}\,\mathrm{d}x$$

where $f(x)$ is a linear function (or constant), and the quadratic in the denominator can be factorised into linear factors.

The idea is to break up the integrand into smaller pieces that are easy to integrate. These pieces depend on the way the denominator factorises.



**Case 1: The denominator factorises into *distinct* linear factors.**

In this case we can rewrite the integrand as:

$$\boxed{\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}.}$$

**Example:** Write $\dfrac{9x+1}{(x-3)(x+1)}$ in the form $\dfrac{A}{x-3} + \dfrac{B}{x+1}$.

$$\frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Want to find A and B

$$\frac{9x+1}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}.$$

Denominators are equal → numerators are equal also.



$$\Rightarrow \quad 9x+1 = A(x+1)+B(x-3)$$
$$= Ax + A + Bx - 3B$$
$$= (A+B)x + (A-3B)$$

Equate coefficients:

$x$: $\quad 9 = A + B \quad$ ①

const: $\quad 1 = A - 3B \quad$ ②

$8 = 4B \quad$ ①-②

$\Rightarrow \quad B = 2 \quad \Rightarrow \quad A = 7$

$$\Rightarrow \quad \frac{9x+1}{(x-3)(x+1)} = \frac{7}{x-3} + \frac{2}{x+1}$$

**Example cont'd:** Hence find $\int \frac{9x+1}{(x-3)(x+1)}\,dx$ .

$$= \int \frac{7}{x-3} + \frac{2}{x+1} \, dx$$

$$= \int \frac{7}{x-3}\, dx + \int \frac{2}{x+1}\, dx$$

let $u = x-3$, $du = dx$ | let $u = x+1$, $du = dx$

$$= 7 \log|x-3| + 2\log|x+1| + C$$



**Example:** $\int \frac{7}{x^2+3x-10}\,dx$

Factorise denominator.

$$\frac{7}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$= \frac{A(x-2) + B(x+5)}{(x+5)(x-2)}$$

$$\Rightarrow \quad 7 = A(x-2) + B(x+5)$$

$$= Ax - 2A + Bx + 5B$$

$$= (A+B)x + (-2A+5B)$$



$x$: $\quad 0 = A + B \quad \Rightarrow \quad A = -B$

const: $\quad 7 = -2A + 5B$ sub.

$$\Rightarrow \quad 7 = -2(-B) + 5B$$

$$= 7B \quad \Rightarrow \quad B = 1, \quad A = -1$$

$$\Rightarrow \int \frac{7}{x^2+3x-10}\,dx = \int \frac{-1}{x+5} + \frac{1}{x-2}\,dx$$

$$= -\log|x+5| + \log|x-2| + c$$

$$= \log\left|\frac{x-2}{x+5}\right| + C$$



**Case 2: The denominator factorises into *repeated* linear factors (i.e. is a perfect square).** In this case we rewrite the integrand as:

$$\boxed{\frac{f(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}\,.}$$

**Example:** $\int \frac{3x+1}{x^2+4x+4}\,dx$

$$\frac{3x+1}{x^2+4x+4} = \frac{3x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$= \frac{A(x+2) + B}{(x+2)^2}$$

$$\Rightarrow \quad 3x+1 = A(x+2) + B$$

$$= Ax + (2A+B)$$

$x$: $3 = A$

const: $1 = 2A + B$

$= 6 + B \Rightarrow B = -5$

$$\Rightarrow \int \frac{3x+1}{x^2+4x+4}\,dx = \int \frac{3}{x+2} - \frac{5}{(x+2)^2}\,dx$$

$$= 3\log|x+2| - \int \frac{5}{(x+2)^2}\,dx$$

let $u = x+2$

$du = dx$

$$= 3\log|x+2| + \frac{5}{x+2} + C$$



**Homework:** $\displaystyle\int \frac{2x-1}{x^2-6x+9}\,dx$

Answer: $2\log_e|x-3| - \frac{5}{x-3} + C$



**Exercise:** Write down the partial fractions decomposition you would use for each of the following:

**(a)** $\dfrac{x+2}{x^2+4x-5} = \dfrac{x+2}{(x+5)(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{x-1}$

**(b)** $\dfrac{1-2x}{x^2+6x+9} = \dfrac{1-2x}{(x+3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2}$

**(c)** $\dfrac{3}{x^2-4} = \dfrac{3}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

**(d)** $\dfrac{4x}{x^2-10x+25} = \dfrac{4x}{(x-5)^2} = \dfrac{A}{x-5} + \dfrac{B}{(x-5)^2}$



**Notes about Partial Fractions**

• The above techniques generalise to cases where the denominator is a product of more than two linear factors. The corresponding partial fractions decompositions are:

$$\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$$

or $$\frac{f(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

These cases are looked at further in Calculus 2.

• If the numerator has degree greater than or equal to that of the denominator, we first need to apply **polynomial long division** to obtain an expression where the numerator has degree smaller than the denominator, before applying the partial fractions method.

**Example:** $\int \frac{2x^3 - 3x^2 - 8x + 24}{x^2 - 4} \, dx$

$$\begin{array}{r l}
 & 2x - 3 \\
x^2 - 4 \,\big) & 2x^3 - 3x^2 - 8x + 24 \\
 & 2x^3 \qquad\;\; - 8x \\
\hline
 & -3x^2 \qquad\; + 24 \\
 & -3x^2 \qquad\; + 12 \\
\hline
 & 12
\end{array}$$

$$\Rightarrow \frac{2x^3 - 3x^2 - 8x + 24}{x^2 - 4} = 2x - 3 + \frac{12}{x^2 - 4}$$

easy to integrate

need partial fractions



$$\frac{12}{x^2 - 4} = \frac{12}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

$$\Rightarrow \quad 12 = A(x+2) + B(x-2)$$

$$= (A+B)x + (2A - 2B)$$

$x$: $\quad 0 = A + B \quad \Rightarrow \quad B = -A$

sub

const: $\quad 12 = 2A - 2B$

$$= 2A - 2(-A)$$

$$= 4A \quad \Rightarrow A = 3, \; B = -3$$



$$\Rightarrow \int \frac{2x^3 - 3x^2 - 8x + 24}{x^2 - 4} \, dx$$

$$= \int 2x - 3 + \frac{3}{x-2} - \frac{3}{x+2} \, dx$$

$$= x^2 - 3x + 3\log|x-2| - 3\log|x+2| + C$$

$$= x^2 - 3x + 3\left(\log|x-2| - \log|x+2|\right) + C$$

$$= x^2 - 3x + 3\log\left|\frac{x-2}{x+2}\right| + C$$



**Homework:** Practice your polynomial long division on these:

(a) $\frac{2x^4 - 6x^3 + 14x^2 - 10x + 19}{x^2 - 3x + 5}$

(b) $\frac{5x^5 + 11x^4 - 3x^3 - 2x^2 - 2x + 1}{x^2 + 2x - 1}$

Answers: (a) $2x^2 + 4 + \frac{2x-1}{x^2-3x+5}$ (b) $5x^3 + x^2 - 1$

## Additional questions

You can now attempt a selection of exercises 9-12 and 15-18 from Chapter 8.5 in the textbook.