Level densities of random matrix difference equations

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Intro

Random matrix theory (RMT) is the study of matrices whose entries are random variables. The probability density for the **entire** matrix is: $P: \ H \to \mathbb{R}_0^+$, where $H \in \mathbb{C}^{m\times n}$.

- choose Hermitian random matrix H ∈ Herm(N) with real eigenvalues E = diag$(E_1, \ldots, E_N)$
- **(averaged) level density**:

$$\bar{\rho}(\lambda) = \langle \rho(\lambda) \rangle = \langle \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - E_j) \rangle,$$

  $\langle \cdots \rangle$ is the average over all random matrices involved
- δ is the **Dirac delta** function

Aim of my project: finding the (averaged) level density $\bar{\rho}_n(\lambda)$ of H_n satisfying the recurrence

$$H_n = A_n H_{n-1} A_n^\dagger + \mathbb{1}_N$$

with $H_0 = \mathbb{1}_N$ and $A_n \in \mathbb{C}^{N\times N}$ random.

Procedure

- **Bijective** relation between **Green's function** $G_n(z)$ and $\bar{\rho}_n(\lambda)$ [1]:

$$\bar{\rho}_n(\lambda) = \frac{1}{\pi} \lim_{\epsilon \searrow 0} \text{Im}(G_n(\lambda - i\epsilon)) \text{ and } G_n(z) = \int_{-\infty}^{\infty} \frac{\bar{\rho}(\lambda) d\lambda}{z - \lambda}$$

- also **bijective** relation between $G_n(z)$ and **R-transform** $R_n(z)$ [2]:

$$R_n(G_n(z)) = z - \frac{1}{G_n(z)}$$

- Formula [3] for calculating $R(z)$: $\{(H_n)_{11}$:(1,1) component of $H_n\}$

$$\lim_{N\to\infty} \frac{1}{N} \frac{d}{dz} \ln \langle e^{Nz(H_n)_{11}} \rangle = R_n(z)$$

- **Laplace's method** [4]: Consider $I = \int_a^b g(t) e^{Nh(t)} dt$.
  Assume $g(c) \neq 0$, $h(t)$ has unique global maximum at c and $h''(c) \neq 0$.

$$\Rightarrow I \sim \frac{\sqrt{2\pi} g(c) e^{Nh(c)}}{\sqrt{-Nh''(c)}} \text{ as } N \to \infty$$

Our random matrix models

- **Unitary invariance**: $P(A_n) = P(UA_n)$ for any unitary matrix $U \in U(N)$
- decompose $v_n = e_1^\dagger A_n = ||v_n|| e_1^\dagger U$
  with $e_1^\dagger = (1, 0, \ldots, 0)$ and $U \in U(N)$
- by unitary invariance

$$\langle e^{NZ(UH_{n-1}U^\dagger)_{11}} \rangle = \langle e^{NZ(H_{n-1})_{11}} \rangle$$

- define $X_j = (H_j)_{11} = e_1^\dagger H_j e_1$, $Y_j = e_1^\dagger A_j A_j^\dagger e_1$
  $\Rightarrow$ reduce to **scalar recurrence**

$$\langle e^{Nz(H_n)_{11}} \rangle = \langle e^{NzX_n} \rangle = \langle e^{Nz(\sum_{i=1}^{n} \prod_{j=i}^{n} Y_j + 1)} \rangle$$

- **unique** maximum $(Y_1^{(0)}, \ldots, Y_n^{(0)}) \Rightarrow$

$$R_n(z) = \sum_{i=1}^{n} \prod_{j=i}^{n} Y_j^{(0)} + 1$$

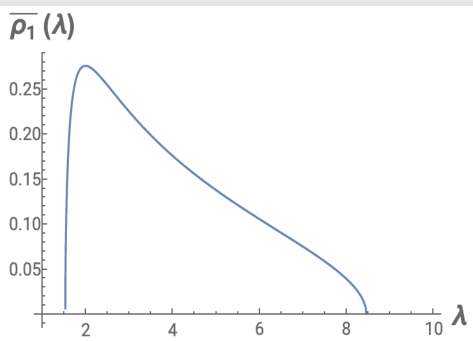
Laguerre

- probability density:

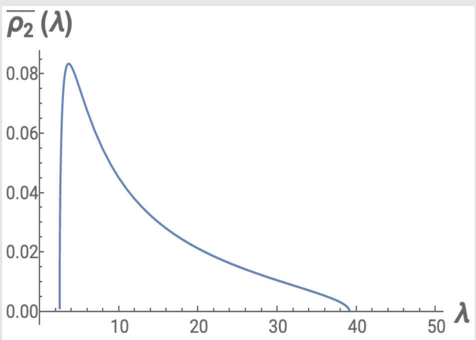
$$P(A) \propto e^{-N\gamma Tr(AA^\dagger)} (det(AA^\dagger))^{N\nu}$$

- recurrence in $G_n(z)$:

$$\frac{(z-1)G_n^2(z) + \nu G_n(z)}{\gamma} = G_{n-1}\left(\frac{\gamma(z-1)}{(z-1)G_n(z) + \nu}\right)$$



n=1, γ=1, ν=2



n=2, γ=1, ν=2

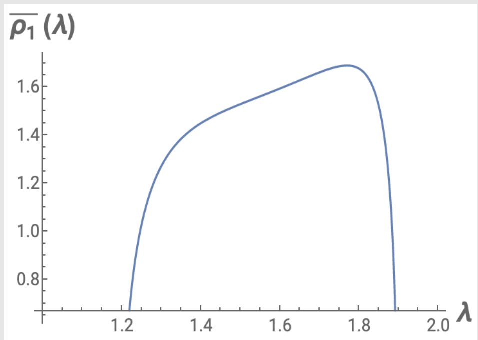
Jacobi

- probability density:

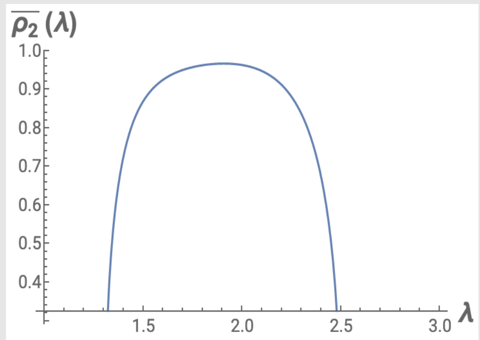
$$P(A) \propto det(\mathbb{1}_N - \gamma AA^\dagger)^{N\mu} (det(AA^\dagger))^{N\nu} \theta(\mathbb{1}_N - \gamma AA^\dagger)$$

- recurrence in $G_n(z)$:

$$\frac{(z-1)G_n^2(z) + \nu G_n(z)}{\gamma} = G_{n-1}\left(\gamma(z-1) + \frac{\gamma(z-1)(1+\mu)}{\nu + (z-1)G_n(z)}\right)$$



n=1, γ=1, ν=3, μ=2



n=2, γ=1, ν=3, μ=2

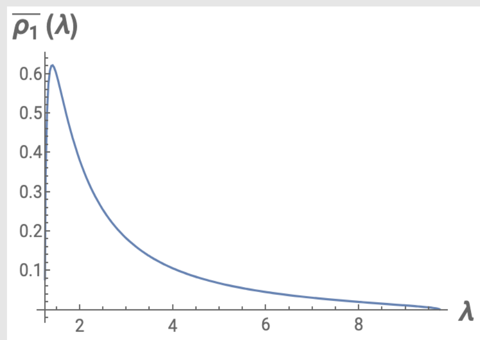
Cauchy-Lorentz

- probability density:

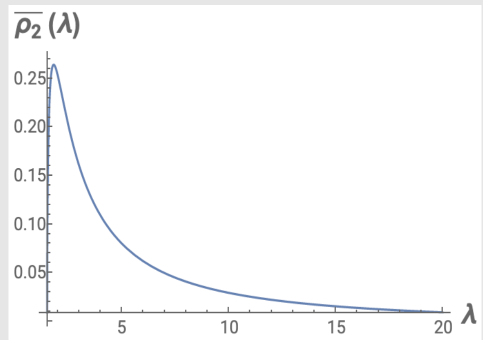
$$P(A) \propto det(\mathbb{1}_N + \gamma AA^\dagger)^{-N(\mu+\nu+2)} (det(AA^\dagger))^{N\nu}$$

- recurrence in $G_n(z)$:

$$\frac{(z-1)G_n^2(z) + \nu G_n(z)}{\gamma} = G_{n-1}\left(-\gamma(z-1) + \frac{\gamma(z-1)(1+\mu+\nu)}{\nu + (z-1)G_n(z)}\right)$$



n=1, γ=1, ν=3, μ=2



n=2, γ=1, ν=3, μ=2

References

[1] Forrester, P. (2010). *Log-Gases and Random Matrices*, London Mathematical Society Monographs Volume 34, Princeton University Press
[2] Speicher, R. (2019). *Lecture Notes on "Free Probability Theory"*, arXiv:1908.08125
[3] Mergny, P & Potters, M. (2020). *Asymptotic behavior of the multiplicative counterpart of the Harish-Chandra integral and the S-transform*, arXiv:2007.09421
[4] Malham, S.J.A. (2005). *An introduction to asymptotic analysis*, Heriot-Watt University