Disaster resilience: augmenting a cycle for planar 3-connectivity

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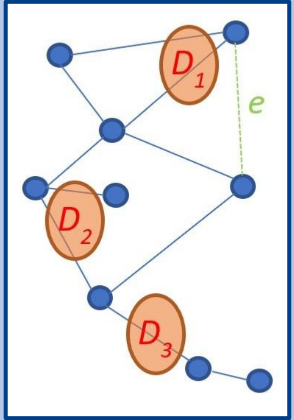
Introduction

How can we best prepare a sewage system across a network of cities to avoid disconnection at the hands of an unanticipated disaster? This conundrum is an instance of the Augmentation for Disaster Resilience problem.

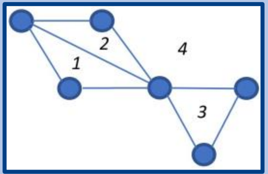


Approaching this problem combinatorically involves:

- ❖ Representing this said network as a graph G on a planar embedding.
- ❖ Adding edges to this to ensure its survivability given a single disaster D of known shape and size, but unknown location.
- ❖ The challenge is doing so optimally, as per a given cost function c.
- ❖ Figure to the left: e makes G resilient to D_1 but not D_2 or D_3.

Augmenting Cycles

Any graph planar G has some number of faces, where a face f is defined as a distinct region bounded by a set of edges which contain no other vertex or edge. The figure on the right has 4 faces.



- ❖ Central to current progress in solving this problem is a lemma pertaining to the independence of the faces:
- ❖ Augmenting edges within some face should not affect how we augment edges within another. In this way, the problem is reduced to augmenting faces, and the subgraphs bordering these faces, individually.
- ❖ One common instance of such a subgraph is a cycle. Thus, focusing on optimally augmenting cycles to increase connectivity is important. The following proof is simplified; click here for the **paper** with the unabridged proof.

Theorem

Let $C = (V, E)$ be an undirected planar cycle embedded in $\mathbb{R}^2$. Let $E' \subseteq V \times V$ be a multiset of embedded edges which are either within or on the boundary of **the same** face of C. A minimum multiset E' that augments C such that $C' = (V, E \cup E')$ is 3-connected and planar is of cardinality $|V| - 1$.

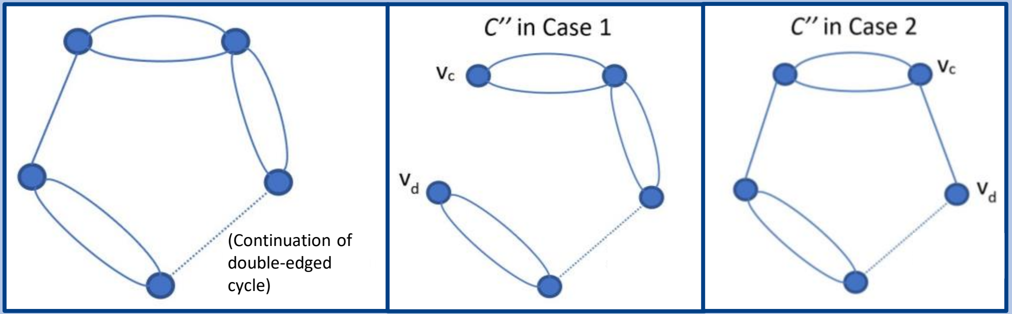
Proof – Upper Bound

Showing that doubling all but one edge achieves 3-connectivity proves |V| - 1 as an upper bound of the cardinality. Menger's theorem is implied here.

Let $E' = E \setminus \{e_1\}$ where $e_1 \in E$, and $C' = (V, E \cup E')$. Suppose we remove edge $e_2 \in E \cup E'$ which joins v_c to v_d, where $v_c, v_d \in V$, from C'. Let $C'' = (V, E \cup E' \setminus \{e_2\})$.

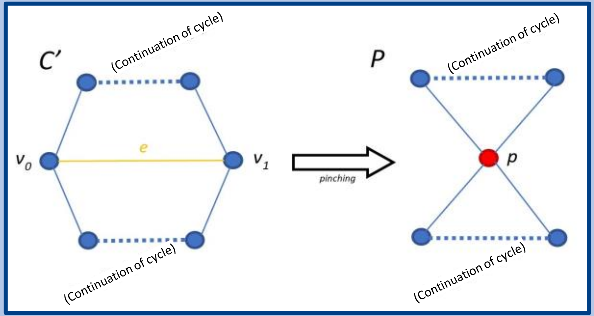
Case 1: e_2 is the only edge in $E \cup E'$ which joins v_c and v_d. Given that C' has all but one edge doubled, e_2 must be the only undoubled edge. What remains after e_2 is removed are two edge disjoint paths traversing the nodes in order $(v_c, \ldots, v_d)$. Therefore, for any two nodes $x, y \in (v_c, \ldots, v_d)$, there are two edge disjoint paths between x and y, and so C'' is 2-connected. Therefore, there exists no e_3 such that removing e_2 and e_3 from C' disconnects it, so C' is 3-connected.

Case 2: There exists another edge $e_4 \in E \cup E' \setminus \{e_2\}$ which joins v_c and v_d. Any node pair (x, y) which was joined by some edge in C trivially remains to be joined by some edge in C', and because the pair v_c and v_d are still connected by e_3, (x, y) remains to be joined by some edge in C''. So C'' remains to be a cycle, and thus 2-connected. Therefore, there exists no e_5 such that removing e_2 and e_5 from C' disconnects it, so C' is 3-connected.

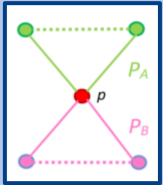


Proof – Lower Bound

Now, proving the upper bound of the minimum multiset's cardinality will ultimately prove the theorem. In constructing E', we first add edge e from v_0 to v_1. Next, we convert the graph C' to P via a manoeuvre named *pinching*. This is done by reducing v_0, e and v_1 down to a singular node p, where the incident edges of p are the union of the edges incident to v_0 and v_1.



Crucially, it is proven that the minimal number of edges needed to augment P to be 3-connected is less than that of C'. Consider the segmented P (right). It is also proven that P is 3-connected iff P_A and P_B are 3-connected. The problem, therefore, is reduced to augmenting each subgraph recursively, until that subgraph is a singular node. Where the number of nodes of C is n, of P_A is j and of P_B is k, the cardinality of a minimum multiset E' is given by the function g:



$$g(n) \geq min\{f(j) + f(k)\} + 1, \text{for some } j + k = n;\ j, k \geq 1$$

With base case: $g(1)=0$. Through strong induction, it is proven that $g(n) \geq n-1$, or $|V|-1$. Therefore, with the upper bound $n-1$ and lower bound at least $n-1$, we have proven the theorem.

References & Acknowledgements:

- ❖ Menger's Theorem, 7/02/2022
- ❖ Gross, Jonathon L, and Yellen, Jay; Graph Theory and Its Applications, CRC Press (December 30,1998). ISBN 0-8493-3982-0.

- ❖ Special thanks to Charl Ras and The University of Melbourne Maths and Statistics Department