Critical points of a spin glass model over a torus

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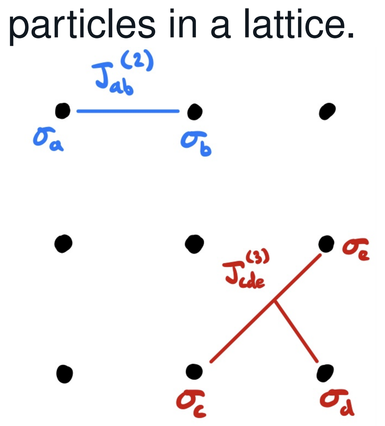
Spin glasses

Roughly speaking, a *spin model* can be thought of as a mathematical model that models interactions between particles in a lattice.



The Hamiltonian H for a general spin model is

$$H(\boldsymbol{\sigma} = \{\sigma_1, ..., \sigma_N\}) = \sum_{ij} J_{ij}^{(2)} \sigma_i \sigma_j + \sum_{ijk} J_{ijk}^{(3)} \sigma_i \sigma_j \sigma_k + ...$$

where $\{J_{i_1,...,i_k}^{(k)}\}$ are constants representing k-point interactions.
A *spin glass* replaces the constants $\{J_{i_1,...,i_k}^{(k)}\}$ with random variables.

A canonical example of a (mean field) spin glass model is the so-called p-spin spherical spin glass model, described by Hamiltonian:

$$H(\boldsymbol{\sigma} = \{\sigma_1, ..., \sigma_N\}) = \sum_{1 \le i_1 < i_2 < ... < i_p \le N} J_{i_1,...,i_p}^{(p)} \sigma_{i_1} ... \sigma_{i_p}$$

where $\{J_{i_1,...,i_p}^{(p)}\}_{1 \le i_1 < i_2 < ... < i_p \le N}$ are independent, identically distributed (i.i.d.) normal (Gaussian) random variables with 0 mean, and the "spins" $\{\sigma_1, ..., \sigma_N\}$ are real numbers satisfying a constraint,

$$\sum_{i=1}^{N} \sigma_i^2 = 1$$

Note that the Hamiltonian H is a random function on the unit sphere S_{N-1}, hence the name spherical spin glass model.

The Kac-Rice formula

Of particular interest is the expected number of critical points of a random function, which we can find by applying the so-called Kac-Rice formula to the gradient field.
Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a random conservative vector field with sufficiently nice realisations. Over some volume V, the Kac-Rice formula says that

$$\mathbb{E}[\#Zeroes(f)] = \int_V \int p_x(0, M) |det(M)| dx dM$$

where:

- $dM = \prod_{i=1}^{N} dM_{ii} \prod_{1 \le i < j \le N} dM_{ij}$ is the flat measure on symmetric matrices
- $p_x(v, M)$ is the joint probability density of the random vector field f and the corresponding Jacobian matrix field at location x.

Spin glass model over a torus

Rather than explore the well-known case of a spin glass over a sphere, we will look at a random function on a torus, i.e. a torodial spin glass model.
The family of models we will focus on is described by Hamiltonian

$$H(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^N} c_{\boldsymbol{k}} \xi_{\boldsymbol{k}} e^{i \boldsymbol{k} \cdot \boldsymbol{x}}$$

where:

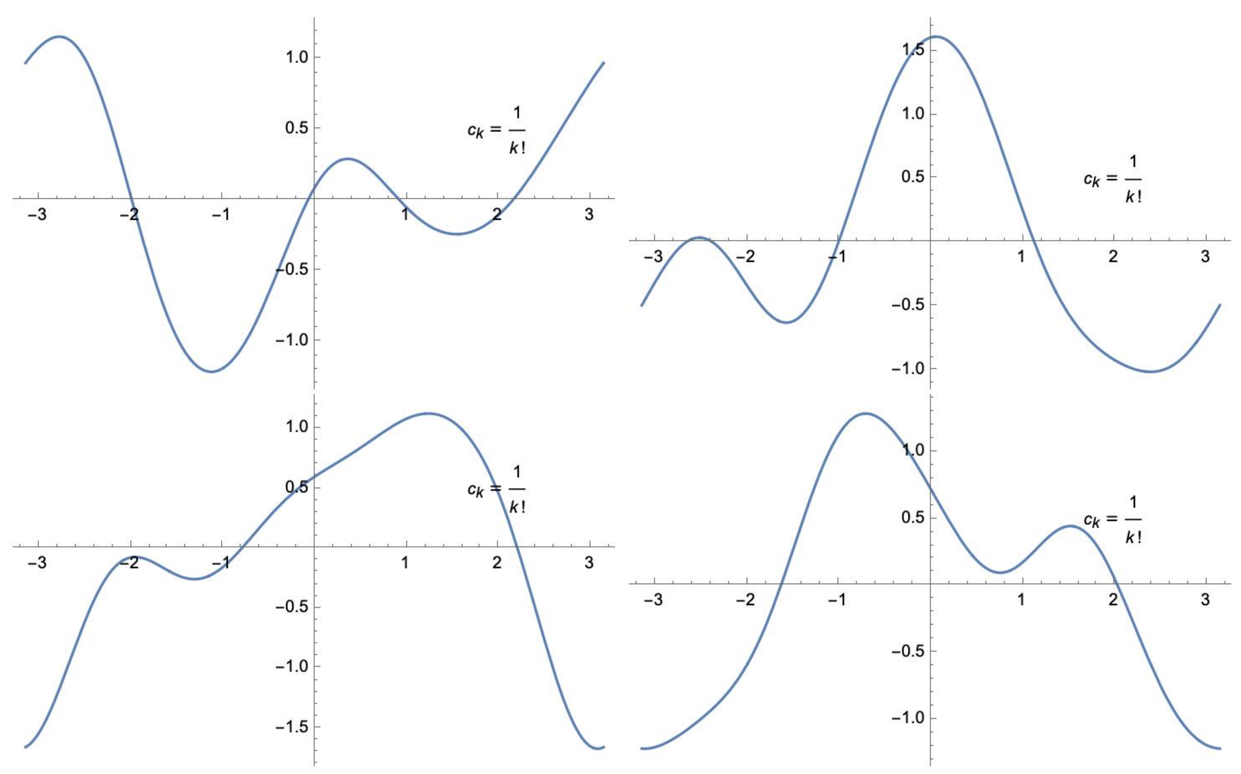
- $\boldsymbol{x} \in T^N = [0, 2\pi)^N$ is the N-dimensional torus.
- $\xi_{\boldsymbol{k}} = X_{\boldsymbol{k}} + iY_{\boldsymbol{k}}$ are independent complex random variables such that $X_{\boldsymbol{k}}$ and $Y_{\boldsymbol{k}}$ are i.i.d. normal random variables with mean 0 and variance $\frac{1}{2}$.
- $(c_{\boldsymbol{k}} \xi_{\boldsymbol{k}})^* = c_{-\boldsymbol{k}} \xi_{-\boldsymbol{k}}$ (this ensures that H is a real valued function).
- $c_{\boldsymbol{k}}$ are constants that depend only on $\|\boldsymbol{k}\|_2$ and decays sufficiently fast.

Examples

We can numerically approximate and graph a few realisations in the $N = 1$ and $N = 2$ case easily.

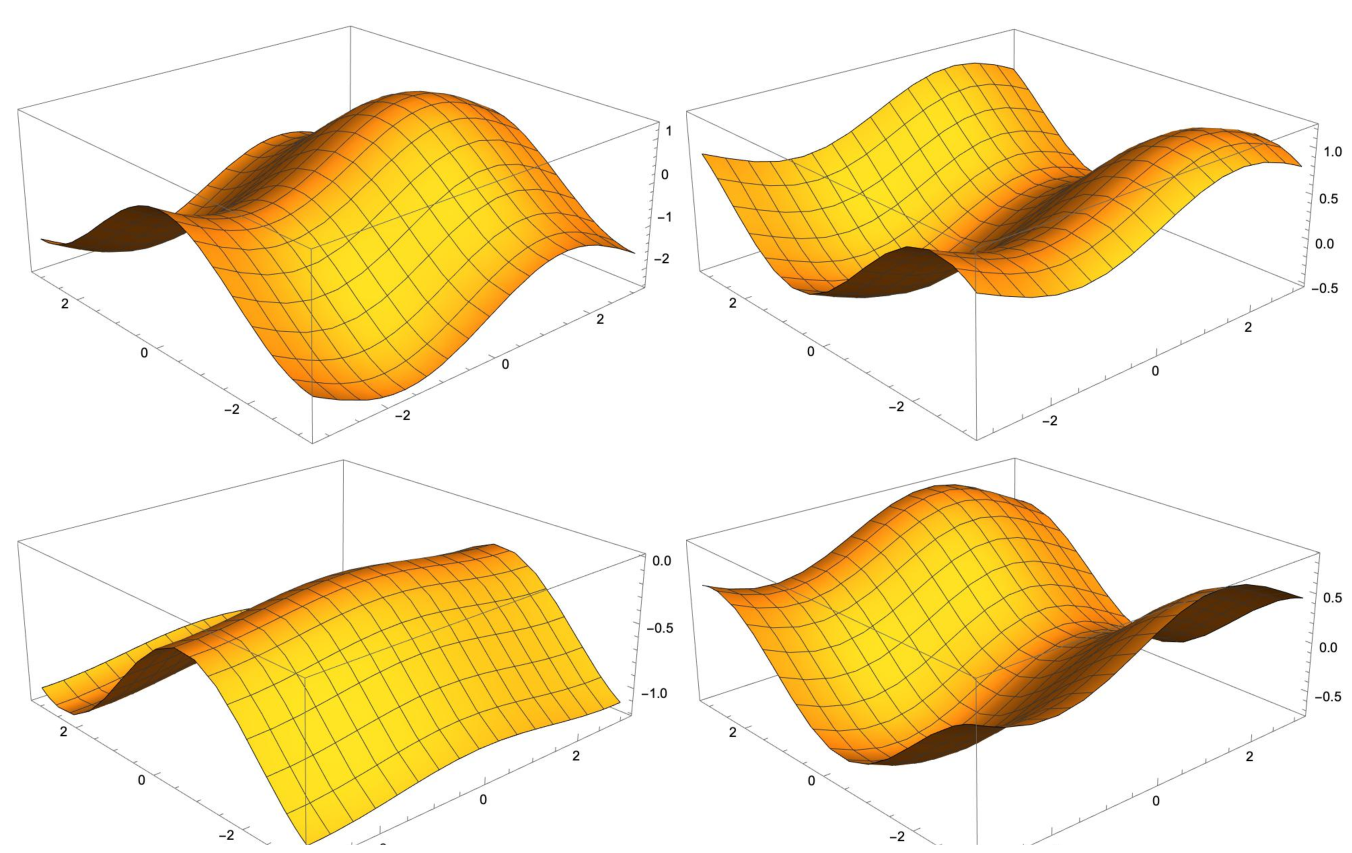
$N = 1$, $c_k = 1/k!$:

$\mathbb{E}[\#Critical\ Points(H)] \approx 3.6854$



$N = 2$, $c_{\boldsymbol{k}} = \frac{1}{1 + \|\boldsymbol{k}\|_2^{20}}$:

$\mathbb{E}[\#Critical\ Points(H)] \approx 4.03008$



Applying the Kac-Rice formula

In applications, often critical points of the Hamiltonian have significance. We can find the expected number of critical points of H by applying the Kac-Rice formula to ∇H.
The Hamiltonian of the torodial spin glass model is a zero mean Gaussian function, thus the joint probability distribution of the *random vector* $\nabla H(\boldsymbol{x})$ and the *random matrix* $Hess(\boldsymbol{x})$, the hessian of $H(\boldsymbol{x})$, is zero mean multivariate normal.
The covariance matrix is organised such that the joint density p is given by

$$p(\boldsymbol{v}, M) = p\left(\boldsymbol{y} = \begin{bmatrix} \boldsymbol{v} \\ svec(M) \end{bmatrix}\right) = \frac{1}{(2\pi)^{\frac{N(N+3)}{4}} \sqrt{det \Sigma}} e^{-\frac{1}{2} \boldsymbol{y}^T \Sigma^{-1} \boldsymbol{y}}$$

where

- $svec(M) = \begin{bmatrix} M_{11} & M_{22} & ... & M_{NN} & M_{12} & M_{13} & ... & M_{N-1,N} \end{bmatrix}^T$
- $\Sigma = \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^T]$

In the torodial model, we have

$$\Sigma = \begin{bmatrix} a\mathbb{1}_N & 0 & 0 \\ 0 & cuu^T + (b-c)\mathbb{1}_N & 0 \\ 0 & 0 & c\mathbb{1}_{\frac{N(N-1)}{2}} \end{bmatrix}, u = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}^T$$

where:

- $a = \frac{1}{N} \sum_{\boldsymbol{k} \in \mathbb{Z}^N} \|\boldsymbol{k}\|_2^2 |c_{\boldsymbol{k}}|^2$
- $b = \frac{1}{N} \sum_{\boldsymbol{k} \in \mathbb{Z}^N} \|\boldsymbol{k}\|_4^4 |c_{\boldsymbol{k}}|^2$
- $c = \frac{1}{N(N-1)} \sum_{\boldsymbol{k} \in \mathbb{Z}^N} (\|\boldsymbol{k}\|_2^4 - \|\boldsymbol{k}\|_4^4) |c_{\boldsymbol{k}}|^2$

The convenient structure of the covariance matrix makes it clear that $p(\boldsymbol{v}, M)$ factorises as

$$p(\boldsymbol{v}, M) = \frac{e^{-\frac{1}{2a} \boldsymbol{v}^T \boldsymbol{v}}}{(2\pi a)^{N/2}} \cdot \frac{e^{-\frac{1}{2} svec(M)^T \sigma^{-1} svec(M)}}{(2\pi)^{\frac{N(N+1)}{4}} \sqrt{det(\sigma)}} = \frac{e^{-\frac{1}{2} \boldsymbol{v}^T \boldsymbol{v}}}{(2\pi a)^{N/2}} \cdot q(M)$$

where $q(M)$ is the density function of $Hess(\boldsymbol{x})$ and $\sigma = \begin{bmatrix} cuu^T + (b-c)\mathbb{1} & 0 \\ 0 & c\mathbb{1} \end{bmatrix}$.
It follows from the Kac-Rice formula that

$$\mathbb{E}[\#Critical\ Points(H)] = \frac{V}{(2\pi a)^{N/2}} \mathbb{E}[|det(M)|]$$

The model is 2π-periodic, so it makes sense to set the volume $V = [0, 2\pi)^N$.
Then $\int_V dx = (2\pi)^N$, so

$$\mathbb{E}[\#Critical\ Points(H)] = \left(\frac{2\pi}{a}\right)^{\frac{N}{2}} \mathbb{E}[|det(M)|]$$

The $N = 1$ case can go a step further, we get

$$\mathbb{E}[\#Critical\ Points(H)] = 2\sqrt{\frac{\sum_{k \in \mathbb{Z}} k^2 c_k^2}{\sum_{k \in \mathbb{Z}} k^4 c_k^2}}$$

Acknowledgements

I would like to express my sincerest gratitude to my supervisor Dr. Jesper Ipsen, for his considerable patience and encouragement over the course of the vacation scholarship program. I would also like to express my gratitude to the School of Mathematics and Statistics for giving me the opportunity to participate.