Towards an Algorithm for the Multi-Source Multi-Sink Steiner Network Problem in the Plane

Harry Xi, Charl Ras

School of Mathematics and Statistics - University of Melbourne

hzxi@student.unimelb.edu.au, cjras@unimelb.edu.au

THE UNIVERSITY OF MELBOURNE

1. Introduction

The '*Minimum Multi-Source Multi-Sink Steiner Network Problem*' (**MMMSNP**) is a directed *optimal interconnection* problem in the plane. It asks, given source set, A and sink set, B, for the shortest directed geometric network, $N = \big(V(N),\, E(N)\big)$, where each $a \in A$ has a path to every $b \in B$. It is closely related to the *undirected Steiner network problem in the plane*; where given a set of n points called terminals, $T = \{t_1, \ldots, t_n\}$, construct a shortest geometric network $N = \big(V(N),\, E(N)\big)$ so all terminals are connected. This has been studied extensively and so, we have a fast, exact algorithm, despite the problem being **NP-hard**. In comparison, the MMMSNP in the plane is relatively unexplored, so we lack an efficient, exact algorithm.

2. Preliminaries

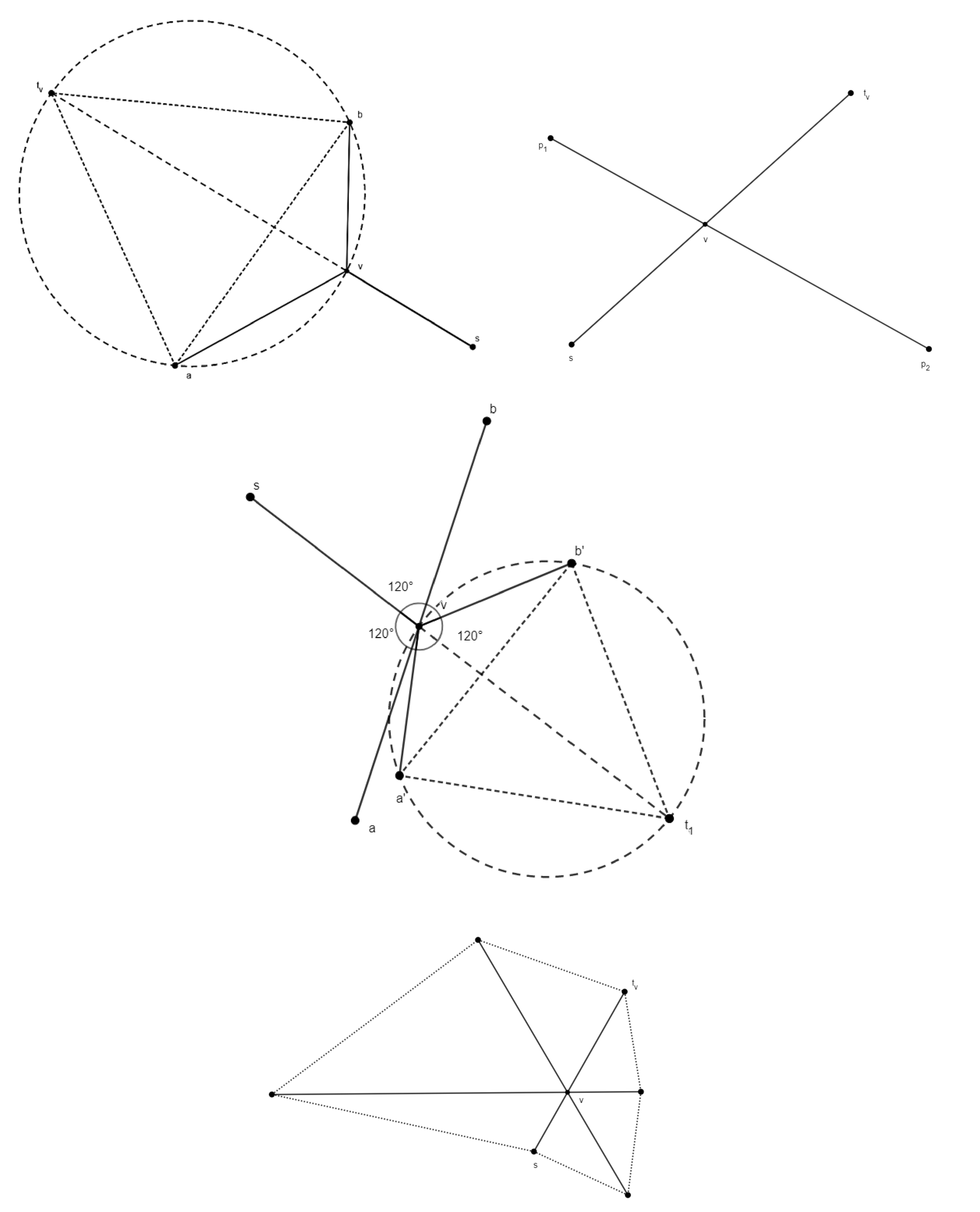
Structural Fundamentals: [1]

- Local geometries of Steiner points have been exhaustively described

Algorithmic Fundamentals: [2]

- GeoSteiner is an algorithm for undirected networks by generating then concatenating full networks
- Point generation: for ordered pairs of equilateral points, generate e_{pq} if $Z(p) \cup Z(q) = \emptyset$ and $|Z(p)| + |Z(q)| < n$
- The first algorithmic framework is seen in [3], by overlaying undirected networks and a mixed-integer program

3. Steiner Arcs for MMMSNP



We generalise Steiner arcs to extend the Melzak algorithm for higher degrees.

4. A Representation for Pseudo-Terminals

Each pseudo-terminal will have a unique representation. For degree 3, 4, 5 and 6 pseudo-terminals;

$$(a, b) \qquad (\{a, b\}, c) \qquad (\{a, b\}, c, d) \qquad (\{a, b\}, \{c, d\}, e)$$

5. Pseudo-Terminals without Pruning

We remove restrictions to achieve an upper bound given by the number of *'properly' parenthesised expressions* with *any* positive number of elements between parentheses or number of solutions to the integer equation,

$$x_1 + \ldots + x_{2ord(p_t)-1} = Z(p_t) - 2 \tag{1}$$

These two options give an upper bound for the number of pseudo-terminals,

$$p(n) < (n+1)! \times M_n \implies p(n) \in O\big(M_n \times (n+1)!\big) \tag{2}$$

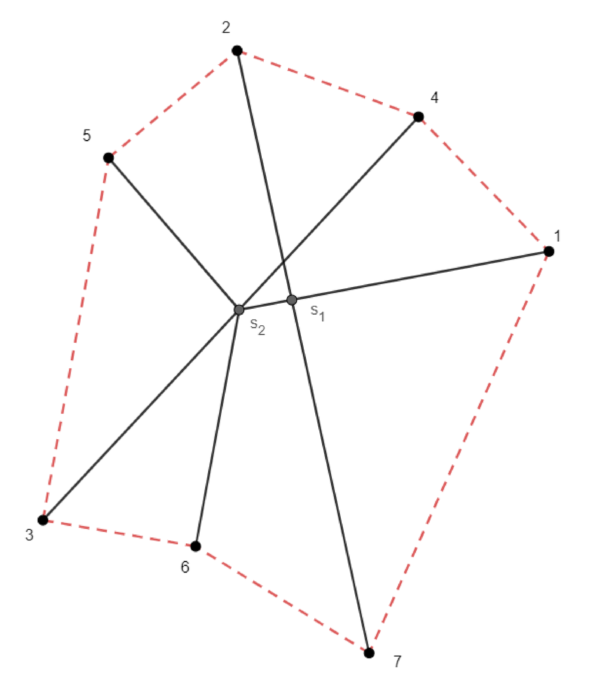
$$p(n) < (n+1)! \times \binom{3n-8}{n-3} \implies p(n) \in O\left(\binom{3n-8}{n-3} \times (n+1)!\right) \tag{3}$$

Where M_n is the nth *Motzkin number*, a sequence with exponential asymptotic behaviour.
By considering pseudo-terminals of order $n-2$, $|Z(p_t)| = n-1$, we bound **below** by $n!$, which is why we need pruning techniques.

6. TSP Approximation

- Take a minimum (A, B)-network, changing single arcs to double arcs. Length represents a full-traversal through Steiner points - at most double the original length
- The shortest Hamiltonian cycle approximation completes the same traversal without Steiner points so by triangle inequality, the TSP solution is 2-optimal
- We use shortest Hamiltonian cycles to prune full Steiner networks

7. TSP Example

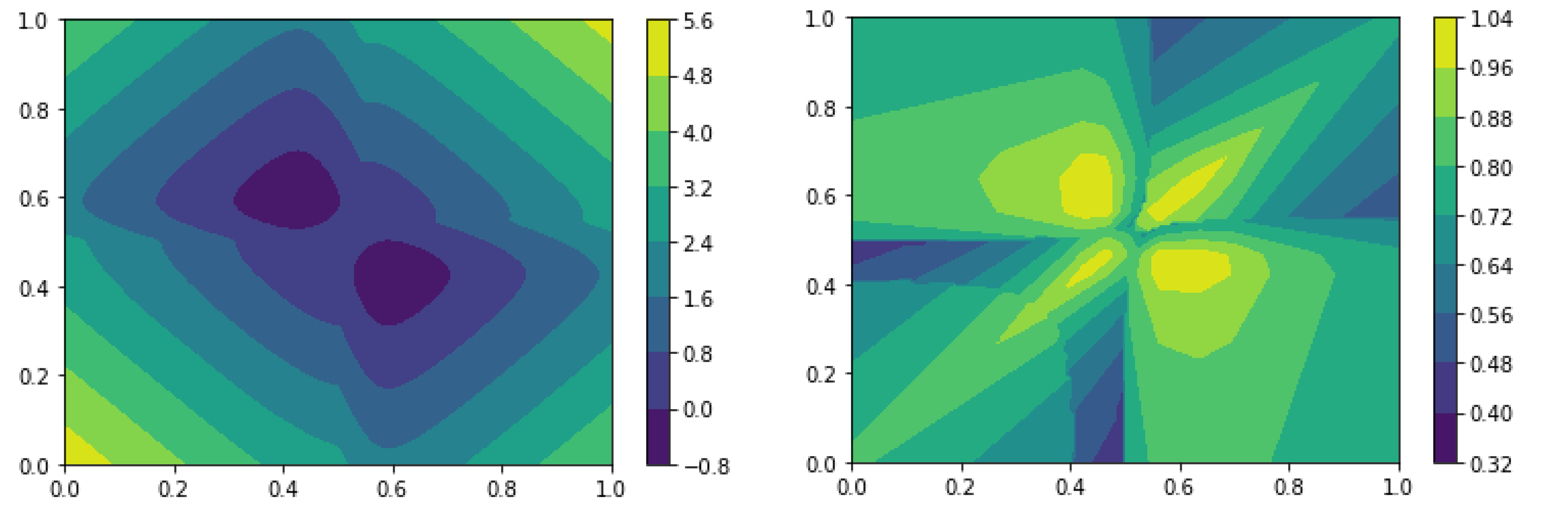


8. Numerical Results

We generate 10000 random pseudo-terminals on 12 terminals, with a uniformly generated location. We use these to test pruning conditions on pseudo-terminals with feasibility of Steiner arcs and angle conditions.

N	proportion pruned	N^*	#pruned/N^*
10000	0.3634	4325	~ 0.84

The rest can be pruned effectively by the tests in [2]. The TSP always gives a cycle in the shape of a *simple polygon* - so we can compare the area bounded by the TSP with the area of the convex hull. In this experiment, we stretch the terminals adjacent to s_1 in the network shown above.



The left diagram shows the difference between the distance of the shortest Hamiltonian cycle and the total length of the network, the right diagram shows the ratio of the area bounded by the shortest Hamiltonian cycle and the convex hull.

9. References

[1] Alastair Maxwell Konrad J. Swanepoel. Shortest directed networks in the plane. *Graphs and Combinatorics*, 36:1457–1475, 2020.

[2] Pawel Winter Martin Zachariasen. Euclidean steiner minimum trees: An improved exact algorithm. *Networks: An International Journal*, 30(3):149–166, 1997.

[3] Alexander Westcott Marcus Brazil Charl Ras. Structural properties of minimum multi-source multi-sink steiner networks in the euclidean plane. *Preprint*, 2023.