Typical entanglement entropy in composite systems with fixed particle number

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Background

The entanglement entropy of a bipartite or two-part quantum system is a characteristic measuring the quantum correlation between two subsystems A and B. Recent research in this field has unveiled the increasing importance of this quantity.

Given a density matrix of a pure quantum state $\rho = |\psi\rangle\langle\psi|$, the reduced density matrix is

$$\rho_A = \operatorname{tr}_B \rho$$

and the entanglement entropy is

$$S_A = -\operatorname{tr}\left(\rho_A \log(\rho_A)\right).$$

This project studies the average entanglement entropy of composite, pure quantum states with a fixed number of particles in the thermodynamic limit.

The System

We consider the general setting of a system with a set of V sites. Each site is described by an identical local Hilbert space $\mathcal{H}_{\text{loc}}$. It decomposes into a direct sum over the number of indistinguishable particles N_{loc} that it holds:

$$\mathcal{H}_{\text{loc}} = \bigoplus_{N_{\text{loc}}} \mathcal{H}_{\text{loc}}^{(N_{\text{loc}})}.$$

The dimension of the Hilbert space $\mathcal{H}_{\text{loc}}^{(N_{\text{loc}})}$ is a positive integer equal to the number of ways to place N_{loc} indistinguishable particles into V distinguishable sites.

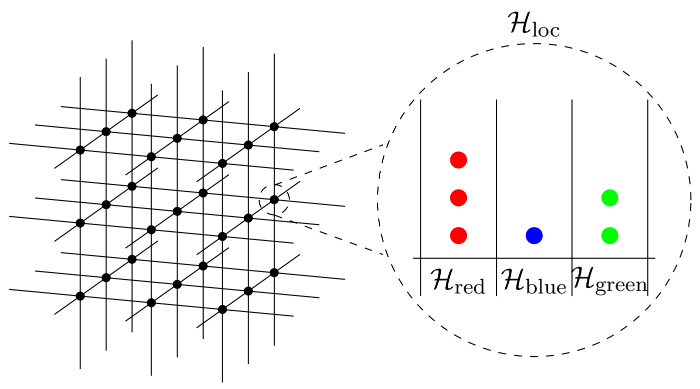

Fig. 1: An example of $\mathcal{H}_{\text{loc}}$, which holds particles in separate sub-sites.

Furthermore, the whole system is described by

$$\mathcal{H} = \bigotimes_{i=1}^{V} \mathcal{H}_i,$$

where $\mathcal{H}_i$ is a copy of $\mathcal{H}_{\text{loc}}$ at the site labelled i.

Dimension

Let us denote $d_N = \dim \mathcal{H}^{(N)}$ and $a_{N_{\text{loc}}} = \dim \mathcal{H}_{\text{loc}}^{(N_{\text{loc}})}$. It turns out that d_N is the z^N-coefficient of the polynomial expansion

$$\begin{aligned} d_N &= [z^N]\left(a_0 + a_1 z + a_2 z^2 + \dots\right)^V \\ &= \frac{1}{2\pi i}\oint_C \frac{\left(\sum_{k=0}^{\infty} a_k z^k\right)^V}{z^N}\frac{\mathrm{d}z}{z}, \end{aligned}$$

where C is a contour enclosing the origin. We introduce $\psi(z) = \log\left(\sum_{k=0}^{\infty} a_k z^k\right) - n\log(z)$, let V approach infinity and evaluate the integral using the saddle-point method, which gives:

$$d_N = \frac{1}{z_0\sqrt{2\pi\psi''(z_0)V}} e^{V\psi(z_0)} + o(1), \qquad (\star)$$

where z_0 is a solution to $\psi'(z) = 0$.

An astounding result! It shows that no matter the structure of $\mathcal{H}_{\text{loc}}^{(N_{\text{loc}})}$, the total dimension d_N scales as $\frac{\alpha(n)}{\sqrt{V}} e^{\beta(n)V}$. The following results apply to all systems of indistinguishable particles with a local Hilbert space structure.

Average Entanglement Entropy

Take a bipartition of the system into subsystem A with V_A sites and subsystem B with $V - V_A$ sites. We fix $n = \frac{N}{V}$ and $f = \frac{V_A}{V}$ as constants, and take the thermodynamic limit $V \to \infty$. Using the main result of Bianchi's paper [1], it follows from $(\star)$ that the average entanglement entropy is:

$$\langle S_A\rangle_N = \beta(n) f V + \frac{f + \log(1-f)}{2} - \frac{\beta'(n)}{\sqrt{-2\pi\beta''(n)}}\sqrt{V}\delta_{f,\frac{1}{2}} + o(1).$$

Note the following:

- it only depends on the exponential scaling of the dimension.
- the leading order term is proportional to the volume of the subsystem, and thus, the volume of the system.
- the constant term is universal throughout **all** systems with indistinguishable particles. We certainly did not expect the leading term to be different, but the constant term to be the same.

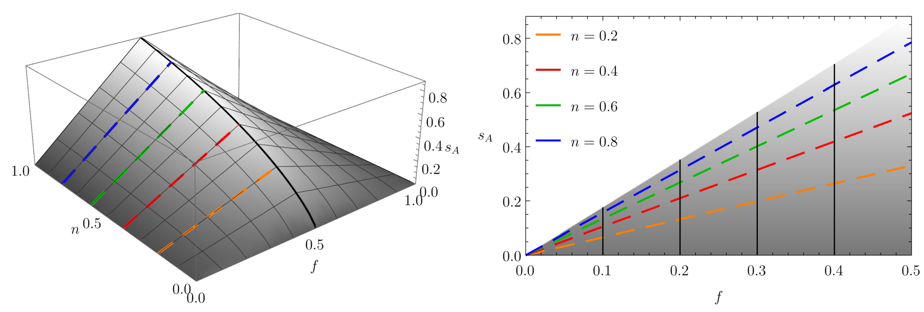

Fig. 2: Leading entanglement entropy $s_A = \lim_{V\to\infty} \frac{\langle S_A\rangle_N}{V}$ for the JCH model. Left: 3-D plot as a function of n and f. Right: Plots at fixed n as functions of f. The coloured lines agree in both plots.

The Jaynes-Cummings-Hubbard (JCH) Model

As an explicit example of a system, take the JCH model. Each site has a photonic cavity, which admits an arbitrary number of photons, and a 2-level atom, which can be thought of as having a maximum of 1 particle. Each non-empty site has two arrangements - either the atom is excited or not - so $a_0 = 1$ and $a_j = 2$ for all $j \neq 0$.

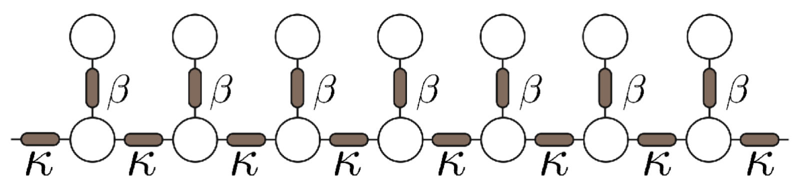

Fig. 3: A diagram of the 1-D JCH model [2], with coupling strengths κ, β.

We find that

$$d_N = \frac{1}{\sqrt{2\pi n\sqrt{1+n^2}V}} e^{V[\operatorname{arcsinh}(n) - 2n\operatorname{arctanh}(n-\sqrt{1+n^2})]} + o(1),$$

which leads to the entropy

$$\begin{aligned} \langle S_A\rangle_N = {} & fV[\operatorname{arcsinh}(n) - 2n\operatorname{arctanh}(n-\sqrt{1+n^2})] + \frac{f+\log(1-f)}{2} \\ & + \sqrt{\frac{2n\sqrt{1+n^2}}{\pi}}\operatorname{arctanh}(n-\sqrt{1+n^2})\sqrt{V}\delta_{f,\frac{1}{2}} + o(1). \end{aligned}$$

References

[1] E. Bianchi and P. Donà. Typical entanglement entropy in the presence of a center: Page curve and its variance. *Physical Review D*, 100(10), 2019.

[2] M. I. Makin. *The Jaynes-Cummings-Hubbard model*. PhD thesis, The University of Melbourne, Melbourne, VIC, 2011.