

Typical volume law entanglement between particle species at fixed N

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Introduction

- ▶ Quantum entanglement quantifies correlations between subsystems that have no classical explanation.
- ▶ A standard way to quantify entanglement is to split a quantum system $\mathcal{H}$ into two subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$ and compute the bipartite entanglement entropy.
- ▶ Most “typical entanglement” results study *geometric* bipartitions (e.g. splitting a lattice into regions).
- ▶ Here we study a novel bipartition: **two particle species** A and B living on the same V -site lattice.

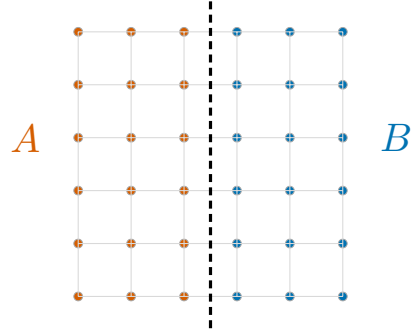


Figure 1. Geometric bipartition (two regions).

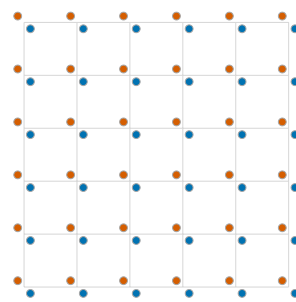


Figure 2. Species bipartition (two species).

Motivation

- ▶ Multi-species systems appear naturally in hybrid light–matter platforms and constrained lattice models (e.g. JCH-type settings).
- ▶ We restrict to systems with a **fixed total particle number** N and look at (Haar-)randomly distributed states.
- ▶ The species are characterised by sequences of natural numbers $(a_{A,j})_j, (a_{B,j})_j \in \mathbb{N}$, where $a_{X,j}$ counts local microstates at excitation level j for species X .

Level j	Degeneracy $a_{A,j}$	Max particles	Interpretation
0	$a_{A,0} = 1$	•	unique vacuum
1	$a_{A,1} = 2$	••	e.g. spin $\uparrow, \downarrow$
2	$a_{A,2} = 3$	•••	(model dependent)

Figure 3. Local states of species A on one lattice site: level j has $a_{A,j}$ microstates.

Aim. Compute an explicit large- V expansion for the typical species entanglement

$$\langle S_A \rangle_N = a(n) V + b(n) \sqrt{V} + c(n) + o(1),$$

and express a, b, c in terms of *combinatorics* of the particle species sequences. We also compute the variance and show self-averaging in the thermodynamic limit.

Method overview

Inputs.

- ▶ Lattice size V and density $n = N/V$.
- ▶ Species data: local microstate counts $a_{A,k}$ and $a_{B,k}$.

Outputs.

- ▶ Coefficients $a(n), b(n), c(n)$ in the typical entropy expansion.
- ▶ Variance scaling showing typicality (self-averaging).

Core engine. Generating functions package the species:

$$\zeta_A(z) = \sum_{k \geq 0} a_{A,k} z^k, \quad \zeta_B(z) = \sum_{k \geq 0} a_{B,k} z^k.$$

1) Specify species data

Choose $\{a_{A,k}\}$ and $\{a_{B,k}\}$.

2) Build generating functions

Use ζ_A, ζ_B to control large- V counting.

3) Fix total N

Decompose into sectors labelled by N_A and compute sector dimensions.

4) Typical entropy formula

Insert sector weights into the Haar-average formula (Page-like).

5) Thermodynamic limit

Saddle/Laplace analysis $\Rightarrow$ extract a, b, c and the variance.

Key results

Main theorem (asymptotic typical entanglement). The typical bipartite entanglement entropy between species satisfies

$$\langle S_A \rangle_N = a(n) V + b(n) \sqrt{V} + c(n) + o(1),$$

where all coefficients are **non-universal** and depend on ζ_A, ζ_B .

Explicit form (branch shown for $\beta_B(n - \bar{n}_A) \leq \beta_A(\bar{n}_A)$).

$$\begin{aligned} \langle S_A \rangle_N &= V \beta_A(\bar{n}_A) - \sqrt{\frac{2V}{\pi \kappa}} |\beta'_A(\bar{n}_A)| \\ &\quad + c(n) + o(1). \\ \kappa &:= -(\beta''_A(\bar{n}_A) + \beta''_B(n - \bar{n}_A)) > 0 \end{aligned}$$

Where $c(n)$ has been computed but is omitted here for brevity.

Takeaway. The leading term is always **volume law**. A $\sqrt{V}$ **correction** appears only at critical densities where the kink ($d_A = d_B$) overlaps with the saddle.

Interpretation.

- ▶ **Volume law:** $a(n) = \beta_A(\bar{n}_A)$ (or $\beta_B(n - \bar{n}_A)$ in the opposite branch).
- ▶ **Species fingerprints:** β_X is determined by ζ_X via $z\zeta'_X(z)/\zeta_X(z) = n_X$.
- ▶ **Constant-order structure:** although not shown, we find that the constant term contains a *species cross-term* Ξ .
- ▶ **Kronecker δ s:** near kinks the contribution can interpolate; we resolve them in the paper.

Case study: Jaynes--Cummings--Hubbard (JCH)

Two species on each site: photons γ and two-level atoms q (excitation number conserved). The single-site generating functions are

$$\zeta_q(z) = 1 + z, \quad \zeta_\gamma(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

The saddle fugacities solve

$$z_{q,0}(n_q) = \frac{n_q}{1-n_q}, \quad z_{\gamma,0}(n_\gamma) = \frac{n_\gamma}{1+n_\gamma}.$$

This gives closed-form entropy densities

$$\beta_q(n_q) = (n_q - 1) \ln(1 - n_q) - n_q \ln n_q, \quad \beta_\gamma(n_\gamma) = (n_\gamma + 1) \ln(1 + n_\gamma) - n_\gamma \ln n_\gamma,$$

and the combined saddle splits $n = \bar{n}_q + \bar{n}_\gamma$ with

$$\bar{n}_q = \frac{(1+n) - \sqrt{1+n^2}}{2}, \quad \bar{n}_\gamma = \frac{(n-1) + \sqrt{1+n^2}}{2}.$$

Punchline: typically $d_\gamma \geq d_q$ at the saddle for all $n \geq 0$, so the kink does *not* overlap with the saddle and the $\sqrt{V}$ term is absent.

Outlook

- ▶ Extend the analysis to bipartitions between species and regions, for example you could compute the entanglement entropy between particles of species A in a specific region of the lattice to everything else.

Acknowledgements and references

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