Adiabatic eigenstates for fermions from complex structures

THE UNIVERSITY OF MELBOURNE

WOST Exploring Quantum Without SpaceTime

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Introduction and motivation

In time-dependent quantum systems, eigenstates are not preserved under unitary time evolution. If, however, the system evolves sufficiently slowly, we can use the **adiabatic theorem** to determine an instantaneous eigenstate that approximately remains an eigenstate at later times. In this project, we develop a construction of these adiabatic eigenstates for fermionic systems in terms of Kähler structures

Fermionic systems & Gaussian states

For a fermionic system with N modes, we can choose a basis of classical Majorana modes, and express our linear observables as $\hat{\xi}^a \equiv (\hat{q}_1, \ldots, \hat{q}_N, \hat{p}_1, \ldots, \hat{p}_N)$. We write a general quadratic Hamiltonian as

$$\hat{H}(t) = \frac{i}{2} h_{ab}(t) \hat{\xi}^a \hat{\xi}^b \tag{1}$$

where h_{ab} is real and antisymmetric. We express the **two-point correlation function** of a state $|\psi\rangle$ as

$$C_2^{ab} = \langle\psi| \hat{\xi}^a \hat{\xi}^b |\psi\rangle = \frac{1}{2}(G^{ab} + i\Omega^{ab}) \tag{2}$$

where Ω^{ab} is an antisymmetric symplectic form that depends on $|\psi\rangle$.

Gaussian states, which arise as the ground states of quadratic Hamiltonians, are central to quantum many-body dynamics as they are **fully characterised by their two-point correlation functions,** C_2^{ab}, allowing for the simplification of many-particle correlation functions via the **Wick-Isserlis theorem**. Importantly, in fermionic systems, a quadratic Hamiltonian will always admit a complete orthonormal basis of Gaussian states.

Kähler structures

In differential geometry, Kähler manifolds are manifolds equipped with three structures, namely a positive-definite metric, a symplectic form, and a complex structure, which we collectively refer to as **Kähler structures**. For a **pure Gaussian state** $|\psi\rangle$, the forms G^{ab} and Ω^{ab} obtained from eq. (2), combined with a complex structure J, where $J^2 = -\mathbb{1}$, form such a **Kähler structure**. The structures are related by

$$J^a{}_b = \Omega^{ac} g_{cb} \tag{3}$$

where $g_{ab} = (G^{ab})^{-1}$. As G is canonical, we can choose an orthonormal basis such that $G = \mathbb{1}$. With G fixed, the dependency of C_2^{ab} on the state will be completely determined J. So, a state $|\psi\rangle$ is **fully characterised by its complex structure,** J.

An instantaneous example

For a system of two fermionic modes, the Hilbert space $\mathscr{H} = \text{span}\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$. In general, Gaussian states of the system are linear combinations of $|00\rangle$ and $|11\rangle$ or $|01\rangle$ and $|10\rangle$. If the system is subject to the stationary Hamiltonian $\hat{H} = \frac{1}{2}(\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1 + \hat{p}_1\hat{q}_2 - \hat{q}_2\hat{p}_1)$, the ground state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is characterised by the complex structure

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

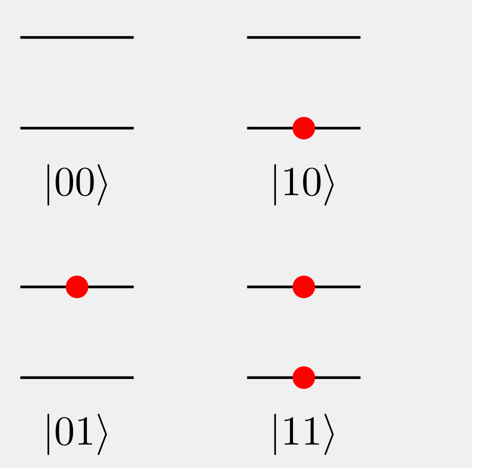

Figure 1. Basis states of a two-mode fermionic system

Interestingly, this is a **Bell state**, an example of a **maximally entangled state**.

Dynamics and construction of J in the time-dependent case

The time evolution of Gaussian states is determined the evolution of their corresponding complex structures. From the Schrödinger equation, we find that $J(t)$ is subject to the dynamical equation

$$\lambda \dot{J}(t) = [K, J(t)] \tag{4}$$

where $K^a{}_b = G^{ac} h_{cb}$. We can slow down the dynamics as necessary by taking $\lambda \ll 1$, and obtain a formal solution by letting $\lambda \to 1$. By expanding $J(t)$ as a perturbation series

$$J(t) = \sum_{k=0}^{\infty} \lambda^k \mathcal{J}_k(t) \tag{5}$$

in eq. (4) and requiring $J(t)^2 = -\mathbb{1}$, we arrive at the system of linear equations for each order of $\mathcal{J}_n$:

$$\text{For } n = 0: \quad [K, \mathcal{J}_0] = 0, \quad \mathcal{J}_0{}^2 = -\mathbb{1} \tag{6}$$

$$\text{For } n \geq 1: \quad [K, \mathcal{J}_n] = \dot{\mathcal{J}}_{n-1}, \quad \{\mathcal{J}_0, \mathcal{J}_n\} = -(\mathcal{J}_1\mathcal{J}_{n-1} + \mathcal{J}_2\mathcal{J}_{n-2} + \ldots + \mathcal{J}_{n-1}\mathcal{J}_1) \tag{7}$$

Proposition. Provided sufficient differentiability of K, we can solve this set of equations order by order to obtain a unique solution for $\mathcal{J}_n$. The proof of this, as outlined here, proceeds by induction:

- **For** $n = 0$: K is antisymmetric, and can thus be **diagonalised into antisymmetric** 2×2 **blocks**. For $\mathcal{J}_0$ to commute with K on each block it must also be block diagonal. For it to be a complex structure we require

$$\mathcal{J}_0 = \left(\tilde{K}^{\vec{n}}\right)^{-1} K, \quad \text{where} \quad \tilde{K}^{\vec{n}} = \bigoplus_{i=1}^{N} (-1)^{n_i} \begin{pmatrix} \omega_i & 0 \\ 0 & \omega_i \end{pmatrix}, \quad \omega_i > 0 \tag{8}$$

  The signs on each block from the factor of $(-1)^{n_i}$ correspond to the occupation number of each mode, which allows us to **define a complex structure for each of the** 2^N **eigenstates** of the system.
- **For** $n \geq 1$: Given that $\mathcal{J}_0, \ldots, \mathcal{J}_{n-1}$ have all been uniquely determined, we obtain the vectorised equation

$$\text{vec}(\mathcal{J}_n) = -\mathcal{S}^{-1}\,\text{vec}((\tilde{K}^{\vec{n}})^{-1}\dot{\mathcal{J}}_{n-1} + \mathcal{J}_1\mathcal{J}_{n-1} + \mathcal{J}_2\mathcal{J}_{n-2} + \ldots + \mathcal{J}_{n-1}\mathcal{J}_1) \tag{9}$$

  where the linear operator $\mathcal{S} = \mathbb{1} \otimes (\tilde{K}^{\vec{n}})^{-1} + ((\tilde{K}^{\vec{n}})^{-1})^{\intercal} \otimes \mathbb{1}$. The existence and uniqueness of $\mathcal{J}_n$ follows directly from $\mathcal{S}$ being **invertible**, as it is then an **injective map**.

Conditions on the existence of a solution

Corollary. The eigenvalues of $\mathcal{S}$ are all pairwise sums of the eigenvalues of $(\tilde{K}^{\vec{n}})^{-1}$, *i.e.*, $\lambda_{\mathcal{S}} = \lambda_i + \lambda_j$, where $\lambda_i = (-1)^{n_i}\omega_i{}^{-1}$. $\mathcal{S}$ will only be singular when the following two conditions are satisfied:

1. K has at least one degeneracy in its spectrum, *i.e.*, $\omega_i = \omega_j$ for some $i \neq j$, and
2. The modes with degenerate eigenvalues have opposite occupation number, *i.e.*, $n_i \neq n_j$, where $n_i, n_j \in \{0, 1\}$.

as this results in a zero eigenvalue of $\mathcal{S}$. In all other cases, $\mathcal{S}^{-1}$ exists, and thus the solution for J_n exists and is unique.

Observation: these conditions are never satisfied for the **ground** and **most excited state** of a system, as they have no mixing of signs. Thus, their **adiabatic eigenstate will always exist**.

Mapping to qubits

For a fermionic system with two modes, h_{ab} will in general have the form

$$h_{ab} \equiv \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix} \tag{10}$$

where each x_i is a real parameter. We find that $\hat{H}$ can be decomposed in block form as

$$\hat{H} = -((\vec{n} \cdot \vec{\sigma}) \oplus (\vec{r} \cdot \vec{\sigma})), \text{ where } \begin{matrix} \vec{n} = \left(\frac{x_3 - x_4}{2}, \frac{x_6 - x_1}{2}, \frac{x_2 + x_5}{2}\right), \\ \vec{r} = \left(-\frac{x_3 + x_4}{2}, -\frac{x_6 + x_1}{2}, \frac{x_2 - x_5}{2}\right) \end{matrix} \tag{11}$$

and where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. As $\hat{H}$ acts on the even and odd sectors of the system independently, we can identify each sector as an independent two-level system, *i.e.*, a **qubit**.

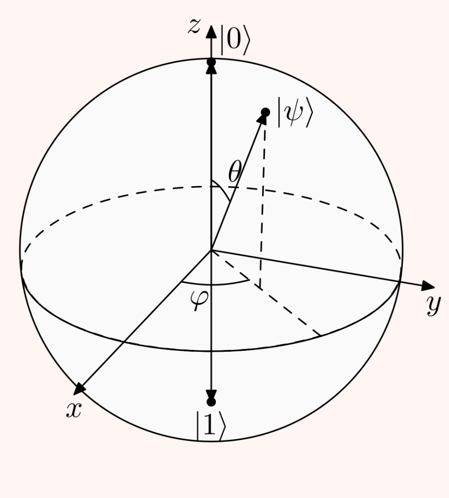

Figure 2. The Bloch sphere: a geometric visualisation of a qubit state.

Due to the isomorphism $\mathbb{C}\mathbf{P}^1 \cong S^2$, we can associate the elements of **projective Hilbert space** with a point on a sphere, called the **Bloch sphere**. For the even sector, a pure Gaussian state will then have the form

$$|\psi\rangle = \cos\frac{\theta}{2}|00\rangle + e^{i\phi}\sin\frac{\theta}{2}|11\rangle, \quad \theta \in [0, \pi], \ \phi \in [0, 2\pi] \tag{12}$$

The vectors $\vec{n}$ and $\vec{r}$, which we also parametrise in spherical coordinates, define the **effective field** applied to each qubit. Instantaneously, the eigenstates of the system are the two points on the sphere intersected by the axis defined by the effective field. The **ground state is aligned** with the field vector, and the **excited state is anti-aligned**. We will examine systems where these vectors **depend adiabatically on time**.

Computing J for the qubit mapping

Consider a system where the effective field vectors are parametrised as

$$\vec{n} = A_0(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), \quad \vec{r} = B_0(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{13}$$

where $\theta = \theta(t)$, $\phi = \phi(t)$ and $A_0 > B_0$, with the initial condition $\theta(0) = 0$, $\phi(0) = \phi_0$. To compute the adiabatic **ground state** at $t_0 = 0$, we first solve the simultaneous equations in eq. (11) to obtain an antisymmetric form for K via h, which we bring into the standard form

$$K = \begin{pmatrix} 0 & A_0 + B_0 & 0 & 0 \\ -(A_0 + B_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & A_0 - B_0 \\ 0 & 0 & -(A_0 - B_0) & 0 \end{pmatrix} \tag{14}$$

via an orthogonal transformation. We then solve eq. (6) and eq. (7) order by order, to obtain

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_1 = \begin{pmatrix} 0 & -\frac{\cos(\phi_0)\theta'(0)}{2A_0} & 0 & -\frac{\sin(\phi_0)\theta'(0)}{2A_0} \\ \frac{\cos(\phi_0)\theta'(0)}{2A_0} & 0 & \frac{\sin(\phi_0)\theta'(0)}{2A_0} & 0 \\ 0 & -\frac{\sin(\phi_0)\theta'(0)}{2A_0} & 0 & \frac{\cos(\phi_0)\theta'(0)}{2A_0} \\ \frac{\sin(\phi_0)\theta'(0)}{2A_0} & 0 & -\frac{\cos(\phi_0)\theta'(0)}{2A_0} & 0 \end{pmatrix} \tag{15}$$

and so forth, where we us eq. (9) each time to obtain the solution for the next order.

Outlook and future research

This construction is a powerful formalism as it can be applied to an **arbitrary number of degrees of freedom**, and unifies various results for both bosons and fermions.

This project is the first step towards a larger work, through which we aim to understand the conditions under which the series expansion for J will truncate, giving us an **exact complex structure**, and to investigate the extension of this method to **fermionic fields in curved spacetime**, where other adiabatic techniques are used to **renormalise** the stress-energy tensor of the field.

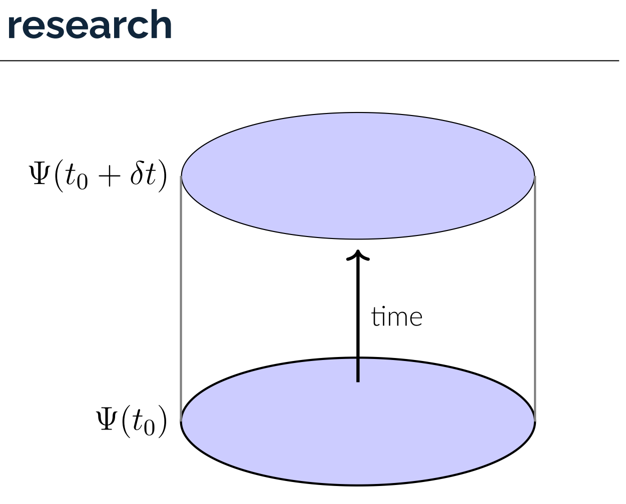

Figure 3. Spacetime diagram showing the adiabatic time evolution of a Dirac field Ψ

Acknowledgements and References

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Figures 1 and 3 were created with TikZ, and Figure 2 was sourced from Wikipedia.

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