Topic 2: Differential Calculus

We now move on to our next topic, differential calculus. This is a fundamental topic in calculus, and is the foundation for our next topics of integral calculus and differential equations. All these topics have major real-world applications, in science, engineering, economics and the natural world.

2.1 Second and higher order derivatives

2.2 Implicit differentiation

2.3 Derivatives of inverse trigonometric functions

2.1.1 Second order derivatives

If f is differentiable at x, then the derivative $f'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ may also be differentiable at x. We may then obtain the **second derivative** of $f(x)$, which is denoted $f''(x)$. There are several different notations which may also be used to denote the second derivative:

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f'(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f(x)\right)$$

Example: If $f(x) = ax^2 + bx + c$, find $f'(x)$ and $f''(x)$.

$f'(x) = 2ax + b$

$f''(x) = 2a$

Example: Let $f(x) = \frac{1}{2}(e^x + e^{-x})$ and $g(x) = \frac{1}{2}(e^x - e^{-x})$.
Find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$. What do you notice?

$f(x) = \frac{1}{2}(e^x + e^{-x})$

$f'(x) = \frac{1}{2}(e^x - e^{-x}) = g(x)$

$f''(x) = \frac{1}{2}(e^x + e^{-x}) = f(x)$

$g(x) = \frac{1}{2}(e^x - e^{-x})$

$g'(x) = \frac{1}{2}(e^x + e^{-x}) = f(x)$

$g''(x) = \frac{1}{2}(e^x - e^{-x}) = g(x)$

2.1.2 Higher order derivatives

Just as we defined the first derivative of f at x to be $f'(x)$, and the second derivative $f''(x)$, it is possible for some functions f to continue taking derivatives $f'''(x)$, $f^{iv}(x)$, and so on. Again, there are several different notations for these higher order derivatives.

If $f^{(n-1)}(x)$ is differentiable at x for some $n \in \mathbb{N}$ then the derivative $f^{(n)}(x)$ is given by

$$f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(n-1)}(x)\right)$$

This is called the n**th derivative** of f.

eg. 6^{th} deriv: $f''''''(x) = f^{vi}(x) = f^{(6)}(x)$

Example: Write down the first five derivatives of $f(x) = \sin x$.

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$

$f^{v}(x) = \cos x$

There are several kinds of problems that we can now solve using our knowledge of higher order derivatives. Essentially, these problems involve repeated differentiation and may involve any of the different techniques we have developed for computing derivatives.

Example: Find a third degree polynomial $P(x)$ such that $P(1) = 1$, $P'(1) = 3$, $P''(1) = 6$ and $P'''(1) = 12$.

Let $P(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow P'(x) = 3ax^2 + 2bx + c$

$P''(x) = 6ax + 2b$

$P'''(x) = 6a$

$P'''(1) = 12$: $12 = 6a \Rightarrow a = 2$

$P''(1) = 6$: $6 = 6a(1) + 2b$

$= 12 + 2b$

$\Rightarrow -6 = 2b \Rightarrow b = -3$

$P'(1) = 3$: $3 = 3a(1)^2 + 2b(1) + c$

$= 6 - 6 + c \Rightarrow c = 3$

$P(1) = 1$: $1 = a(1)^3 + b(1)^2 + c(1) + d$

$= 2 - 3 + 3 + d \Rightarrow d = -1$

$\Rightarrow P(x) = 2x^3 - 3x^2 + 3x - 1$

Example: If $f(x) = \dfrac{1}{x}$, compute f', f'', f''' and f^{iv} and use these to write down a formula for the nth derivative, $f^{(n)}(x)$.

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = +2x^{-3}$

$f'''(x) = -2 \cdot 3\, x^{-4}$

$f^{iv}(x) = \underbrace{+}_{\text{sign}}\ \underbrace{2 \cdot 3 \cdot 4}_{\text{coeff.}}\ x^{\overbrace{-5}^{\text{power}}}$

Look for a pattern:

derivative	sign	coeff.	power
0	+	1	−1
1	−	1	−2
2	+	2	−3
3	−	2·3	−4
4	+	2·3·4	−5
n	$(-1)^n$	$n!$	$-(n+1)$

$\rightarrow f^n(x) = (-1)^n\, n!\, x^{-(n+1)}$

Additional questions

You can now attempt a selection of problems from 39 - 40 in Chapter 3.3 from the textbook.

2.2 Implicit Differentiation [Chapter 4.1]

Usually, to find the derivative $\frac{dy}{dx}$ we are given or deduce y as a function of x and apply our knowledge of differentiation.

Warning! Sometimes it can be impossible to write down an expression for $y = f(x)$.

Example: Given $x^2 - xy + y^4 = 5$, what is $y(x)$?

$$-xy + y^4 = 5 - x^2$$

$$y(-x + y^3) = 5 - x^2$$

$$y = \; ??$$

This is not immediately obvious and motivates the idea of implicit differentiation.

First, we consider a much simpler expression from which we can immediately write down an expression for the derivative $\frac{dy}{dx}$. However, for this first time, we will take a much longer approach to show that there is an alternative way to arrive at the same result. The new approach is called **implicit differentiation** and allows us to write down the derivative of complicated expressions.

Consider

$$x^2 y = 1 \tag{2}$$

In this case we could easily deduce

$$y(x) = \frac{1}{x^2} \qquad \text{and so} \qquad \frac{dy}{dx} = -\frac{2}{x^3}$$

provided $x \neq 0$.

Let's now look at another approach. Consider (2). Even though we have not written $y = f(x)$ explicitly, progress can be made by assuming that $y = f(x)$, where f is not necessarily known. That is, we assume *implicitly* that y is a function of x.

After making this assumption, we can differentiate both sides of (2) with respect to x.

$$\frac{d}{dx}\left(x^2 y\right) = \frac{d}{dx}(1)$$

Applying the product rule to the left hand side gives

$$x^2 \frac{d}{dx}(y) + y\frac{d}{dx}\left(x^2\right) = 0$$

$$\Rightarrow \qquad x^2 \frac{dy}{dx} + y \cdot 2x = 0$$

Then rearranging to make $\frac{dy}{dx}$ the subject gives:

$$\frac{dy}{dx} = -\frac{2y}{x}$$

This expression is in a different form to what we're used to.

Previously, we have written $\frac{dy}{dx}$ in terms of x only. Notice that here we have written $\frac{dy}{dx}$ in terms of both x and y.

For this simple problem, where we can obtain y as a function of x, we can substitute this expression into $\frac{dy}{dx}$, to obtain the derivative as a function solely of x:

$$\frac{dy}{dx} = -\frac{2}{x}y = -\frac{2}{x}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$$

which is the same as what we found the quick way.

Note: Usually when we use implicit differentiation, it will not be possible to obtain $\frac{dy}{dx}$ as a function solely of x, so we normally keep $\frac{dy}{dx}$ written in terms of both x and y.

The above example illustrates the general procedure for finding derivatives by implicit differentiation.

General procedure for implicit differentiation

1. Start with an equation involving both x and y
2. Suppose that y is *implicitly* a function of x
3. Take the derivative of each side with respect to x
4. Use the usual differentiation rules to simplify each side
5. Rearrange to solve for $\frac{dy}{dx}$

Let's take a closer look...

Let's look more closely at what's happening, through a different example.

Example: If $y^3 = x$, find $\dfrac{dy}{dx}$ by implicit differentiation.

1. We have $y^3 = x$.
2. Suppose that y is implicitly a function of x, $y = y(x)$.
3. $\dfrac{d}{dx}(y^3) = \dfrac{d}{dx}(x)$

Hmm... on the left hand side we need to differentiate y^3 with respect to x (not y). How can we do this?

Recall that we have assumed that y is implicitly a function of x. So y^3 is essentially a *function of a function.* The 'cube' function is applied to y, which itself is a function of x.

We know how to differentiate such composite functions – apply the chain rule! So:

$$\begin{aligned}\frac{d}{dx}(y^3) &= \frac{d}{dy}(y^3)\cdot\frac{dy}{dx}\\ &= 3y^2\cdot\frac{dy}{dx}\end{aligned}$$

The $3y^2$ is the derivative of the outer function and the $\frac{dy}{dx}$ is the derivative of the inner function.

So... back to step 4:

4. $$\frac{d}{dx}(y^3) = \frac{d}{dx}(x)$$

$$\Rightarrow \quad 3y^2\cdot\frac{dy}{dx} = 1$$

5. $$\frac{dy}{dx} = \frac{1}{3y^2}$$ ... Done!

Example: If y is considered to be implicitly a function of x, find the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(y^5) = \frac{d}{dy}(y^5)\frac{dy}{dx} = 5y^4\frac{dy}{dx}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(y)) = \frac{d}{dy}(\sin(y))\frac{dy}{dx} = \cos(y)\frac{dy}{dx}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(e^y) = \frac{d}{dy}(e^y)\frac{dy}{dx} = e^y\frac{dy}{dx}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(\log(y)) = \frac{d}{dy}(\log(y))\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$

See the pattern?

Differentiate with respect to y, then multiply by $\frac{dy}{dx}$

Let's return to our original problem. Here we cannot solve for y as a function of x, so implicit differentiation is the only way to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example: If $x^2 - xy + y^4 = 5$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{d}{dx}(x^2 - xy + y^4) = \frac{d}{dx}(5)$$

$$\Rightarrow \frac{d}{dx}(x^2) - \frac{d}{dx}(\overset{u \cdot v}{x \cdot y}) + \frac{d}{dx}(y^4) = \frac{d}{dx}(5)$$

$$\Rightarrow 2x - \left(\overset{u}{x} \cdot \overset{v'}{\frac{d}{dx}(y)} + \overset{v}{y} \cdot \overset{u'}{\frac{d}{dx}(x)}\right) + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow 2x - \left(x\frac{dy}{dx} + y \cdot 1\right) + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow 2x - x\frac{dy}{dx} - y + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(4y^3 - x) = y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 2x}{4y^3 - x}$$

Homework: If $\cos(y) = x^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{\sin(y)}$

Example: If $\sin(\log(y)) = \frac{x}{y}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

quotient rule:

$$\frac{y \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(y)}{y^2}$$

$$\frac{d}{dx}(\sin(\log(y))) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{d}{dy}(\sin(\log(y)))\frac{dy}{dx} = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$\Rightarrow \cos(\log(y)) \cdot \frac{1}{y}\frac{dy}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\Rightarrow y\cos(\log(y))\frac{dy}{dx} = y - x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(y\cos(\log(y)) + x \right) = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{y\cos(\log(y)) + x}$$

Example: Find the equation of the tangent to the curve $y^4 - x^4 = 15$ at the point $(1, 2)$.

Slope of tangent line is given by $\frac{dy}{dx}$

$\rightarrow$ Find $\frac{dy}{dx}$:

$$\frac{d}{dx}(y^4 - x^4) = \frac{d}{dx}(15)$$

$$\frac{d}{dx}(y^4) - \frac{d}{dx}(x^4) = \frac{d}{dx}(15)$$

$$\frac{d}{dy}(y^4)\frac{dy}{dx} - 4x^3 = 0$$

$$4y^3\frac{dy}{dx} - 4x^3 = 0$$

$$4y^3\frac{dy}{dx} = 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3}{y^3}$$

At the point $(1,2)$:

$$\frac{dy}{dx} = \frac{1^3}{2^3} = \frac{1}{8}$$

$\Rightarrow$ Equation of tangent line is

$$y - y_0 = m(x - x_0)$$

$$y - 2 = \frac{1}{8}(x - 1)$$

Homework: Find the equation of the tangent to the hyperbola $x^2 - y^2 = 9$ at the point $(5, 4)$.

Answer: $y - 4 = \frac{5}{4}(x - 5)$

Additional questions

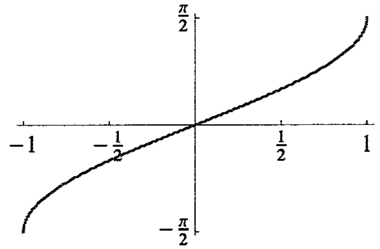
You can now attempt a selection of problems from 1 - 20 in Chapter 4.1 from the textbook.

2.3 Derivatives of inverse trigonometric functions [Chapter 4.3]

So far we haven't discussed the derivatives of the inverse trigonometric functions, arcsin, arccos and arctan. Now, with our knowledge of implicit differentiation, we are able to find these.

2.3.1 Derivative of arcsin

Recall the arcsin function, which has domain $[-1,1]$ and range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$:



We would like to find the derivative $\frac{dy}{dx}$ of

$$y = \arcsin(x)\ ,$$

where $x \in [-1,1]$, $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

We can rearrange this to give

$$\sin(y) = x \qquad (3)$$

We can then use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{d}{dx}\big(\sin(y)\big) = \frac{d}{dx}(x)$$

$$\Rightarrow \quad \cos(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)} \qquad (4)$$

So we have found $\frac{dy}{dx}$ in terms of y but would like to express it as a function of x.

From (3) we know $\sin(y) = x$, but we need $\cos(y)$ in terms of x. Since $\sin^2(y) + \cos^2(y) = 1$, we have

$$\cos(y) = \sqrt{1-\sin^2(y)}$$

(where the positive root is taken since $y \in [-\frac{\pi}{2},\frac{\pi}{2}]$, so $\cos(y) \geq 0$.)

Substituting this in (4) gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cos(y)}\\ &= \frac{1}{\sqrt{1-\sin^2(y)}}\\ &= \frac{1}{\sqrt{1-x^2}} && \text{from (3)}\end{aligned}$$

So we have found the derivative of arcsin!

$$\boxed{\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}\,, \qquad \text{for } -1 < x < 1}$$

We can add this to our list of standard integrals. However, if you can't remember this formula, remember you can always derive it in the above way.

Example: Use the derivative of arcsin and the chain rule to find

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\arcsin(5x)\big)$

$$= \frac{1}{\sqrt{1-(5x)^2}} \times 5 = \frac{5}{\sqrt{1-25x^2}}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\arcsin(x^3)\big)$

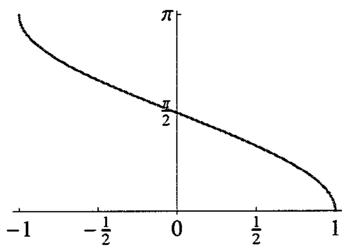
$$= \frac{1}{\sqrt{1-(x^3)^2}} \times 3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

Homework: Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\arcsin(2x+1)\big)$.

Answer: $\frac{2}{\sqrt{1-(2x+1)^2}}$

2.3.2 Derivative of arccos

Recall the arccos function, which has domain $[-1,1]$ and range $[0,\pi]$:



We again use implicit differentiation to find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of

$$y = \arccos(x)\,,$$

where $x \in [-1,1]$, $y \in [0,\pi]$.

First rewrite $y = \arccos(x)$ as

$$\cos(y) = x \qquad (5)$$

Then we find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\cos(y)\big) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

$$\Rightarrow \quad -\sin(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}} \qquad \text{from (5)}$$

Again, we have taken the positive square root above since $y \in [0,\pi]$, so $\sin(y) \geq 0$.

So the derivative of arccos is:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big(\arccos(x)\big) = \frac{-1}{\sqrt{1-x^2}}\,, \qquad \text{for } -1 < x < 1}$$

Again, it may be useful to remember this formula, but if not you can derive it like above.

Example: If $y = \arccos\left(\frac{x}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \times \frac{1}{3}$$

$$= \frac{-1}{\sqrt{\frac{9}{9}-\frac{x^2}{9}}} \times \frac{1}{3}$$

$$= \frac{-1}{\cancel{\frac{1}{3}}\sqrt{9-x^2}} \times \cancel{\frac{1}{3}}$$

$$= \frac{-1}{\sqrt{9-x^2}}$$

Example: If $y = \arccos\left(\sqrt{x}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{-1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

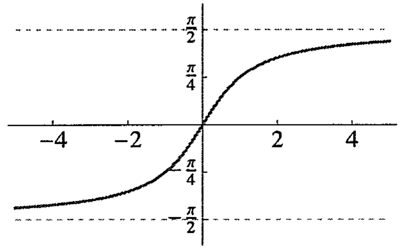
$$= \frac{-1}{2\sqrt{x}\sqrt{1-x}}$$

Homework: Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos(e^x)\right)$.

Answer: $\frac{-e^x}{\sqrt{1-e^{2x}}}$

2.3.3 Derivative of arctan

Recall the arctan function, which has domain $\mathbb{R}$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:



We now work out the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of

$$y = \arctan(x),$$

where $x \in \mathbb{R}$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$y = \arctan(x)$$

$$\Rightarrow \tan y = x$$

$$\Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Know $\tan y$ in terms of x.

Need $\sec^2 y$ in terms of x.

$$\tan^2 y + 1 = \sec^2 y$$

$$\Rightarrow x^2 + 1 = \sec^2 y$$

So $$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$= \frac{1}{1+x^2}$$

So the derivative of arctan is:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big(\arctan(x)\big) = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}}$$

Example: If $f(x) = \arctan\left(\frac{2x-1}{5}\right)$, find $f'(x)$.

$$f'(x) = \frac{1}{1 + \left(\frac{2x-1}{5}\right)^2} \times \frac{2}{5}$$
$$= \frac{1}{\frac{25}{25} + \frac{(2x-1)^2}{25}} \times \frac{2}{5}$$
$$= \frac{25}{25 + (2x-1)^2} \times \frac{2}{5}$$
$$= \frac{10}{25 + (2x-1)^2}$$

Homework: Find $\frac{\mathrm{d}}{\mathrm{d}x}\big(\arctan(x^5)\big)$.

Answer: $\frac{5x^4}{1+x^{10}}$

Example: If $f(x) = (1+x^2)\arctan(x)$, find $f'(x)$.

$$f'(x) = (1+x^2) \cdot \frac{1}{1+x^2} + \arctan x \cdot 2x$$
$$= 1 + 2x \arctan x$$

Example: If $f(x) = \sqrt{\arcsin(x)}$, find $f'(x)$.

$$f'(x) = \frac{1}{2\sqrt{\arcsin x}} \times \frac{1}{\sqrt{1-x^2}}$$
$$= \frac{1}{2\sqrt{\arcsin x}\,\sqrt{1-x^2}}$$

Additional questions

You can now attempt a selection of exercises from 33-48 in Chapter 4.3 in the textbook.