The Jones polynomial in terms of braid group representations

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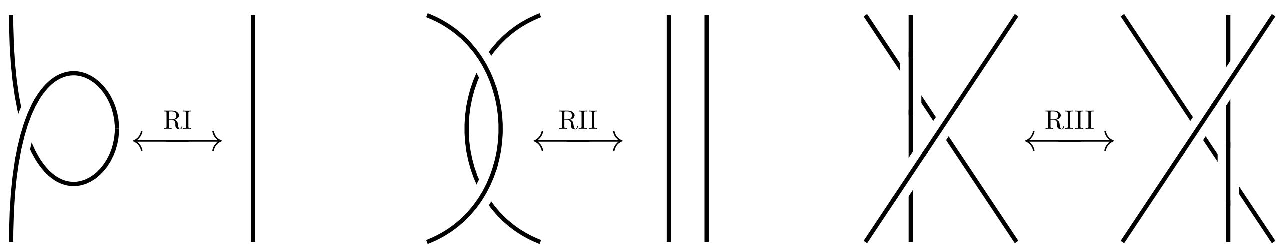
Overview

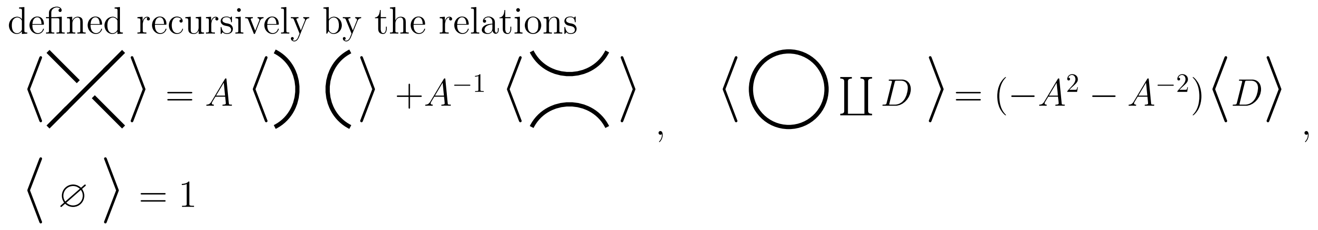
A knot is a simple closed curve in $\mathbb{R}^3$; we draw a knot diagram by projecting a knot onto $\mathbb{R}^2$. A link is a disjoint union of knots, each considered a component of the link. Two knots are isotopic if one can be continously deformed into each other without 'breaking' the knot, or equivalently, if they are related to another by a sequence of Reidemeister moves (RI, RII, RIII):



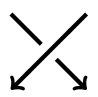
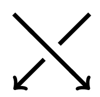
A knot invariant assigns a value to a knot such that if two knots are isotopic, they have the same value under the invariant. We study the Jones polynomial knot invariant and outline a construction of the Jones polynomial via representations of braid groups.

The Kauffman bracket and Jones polynomial

For a link diagram D, the Kauffman bracket $\langle D\rangle$ is a Laurent polynomial in A defined recursively by the relations



where the diagrams in the first relation correspond to links identical everywhere except around a crossing, and the diagram of the second relation corresponds to the nonintersecting union of an unknot and D.
By orienting each component of a link D, we define the sign of a crossing and the writhe $w(D)$ of an oriented link D as follows:

 = positive crossing,  = negative crossing,

$$w(D) = (\#\text{positive crossings}) - (\#\text{negative crossings}).$$

Then by modifying the Kauffman bracket to be invariant under the Reidemeister moves, we obtain the Kauffman bracket invariant $(-A^3)^{-w(D)}\langle D\rangle$ for an oriented link. The Jones polynomial $V_L(t)$ can be defined in terms of the Kauffman bracket by dividing the value of the Kauffman bracket invariant by $(-A^2 - A^{-2})$ and substituting $A^2 = t^{-1/2}$.
The skein relation of the Jones polynomial is

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$$

where L_+, L_- and L_0 are link diagrams identical everywhere except for an area around a crossing, replaced by a positive crossing, negative crossing and no crossing respectively.

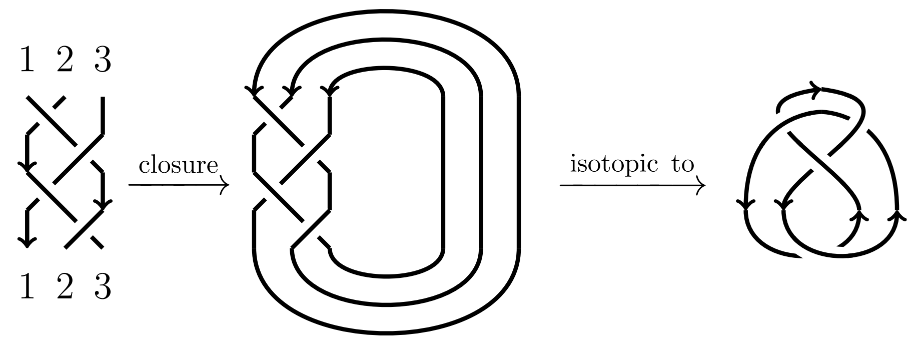
The braid group and Markov moves

A braid is a set of n strands in $\mathbb{R}^2 \times \{0,1\}$, each with distinct endpoints: one in $\{1,2,...,n\}\times\{0\}\times\{0,1\}$ and one in $\{1,2,...,n\}\times\{1\}\times\{0,1\}$. Any braid can be expressed as an element of the braid group of n strands B_n, with presentation

$$\text{generators}: \sigma_1, \sigma_2, ..., \sigma_{n-1}$$

$$\text{relations}: \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

where σ_i denotes a positive crossing between the i-th and $(i+1)$-th strands. The closure of a braid can be formed by connecting the ends $(i,1,0)$ and $(i,1,1)$; by Alexander's theorem, any (oriented) link is isotopic to the closure of a braid. For example, the closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$ is isotopic to the figure-eight knot as shown:



The closures of braids a and b are isotopic to each other if and only if braids a and b are related to each other by the Markov moves (MI, MII):

$$\text{MI}: ab \longleftrightarrow ba, \quad \text{MII}: b \longleftrightarrow b\sigma_n^{\pm 1},$$

where $a, b \in B_n$ and $b\sigma_n^{\pm1} \in B_{n+1}$.

Representations of the braid group

For a vector space V over $\mathbb{C}$, we consider the representation $\psi_n : B_n \to \mathrm{End}(V^{\otimes n})$ given by $\psi_n(\sigma_i) = (id_V)^{\otimes(i-1)} \otimes R \otimes (id_V)^{\otimes(n-i-1)}$, where $R \in \mathrm{End}(V\otimes V)$. From the presentation of the braid group, ψ_n must satisfy

$$\psi_n(\sigma_i)\psi_n(\sigma_j) = \psi_n(\sigma_j)\psi_n(\sigma_i),\ |i-j| \geq 2,$$

$$\psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) = \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}).$$

From these relations, R must satisfy the Yang-Baxter equation

$$(id_V \otimes R)(R\otimes id_V)(id_V\otimes R) = (R\otimes id_V)(id_V\otimes R)(R\otimes id_V).$$

Representations of the braid group (cont.)

We consider the case where V is a 2-dimensional vector space and R is of the form

$$R = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f\end{bmatrix}.$$

From the Yang-Baxter equation, if $b = 0$ and $e \neq 0$, after normalising the entries in terms of t we obtain the R matrices

$$R = \begin{bmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & t^{1/2}-t^{3/2} & 0 \\ 0 & 0 & 0 & t^{1/2}\end{bmatrix}, \begin{bmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-1/2}-t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2}\end{bmatrix}.$$

which correspond to the Jones and Alexander polynomials respectively.
By taking the trace of $\psi_n(b)$ where b is a braid, we try to construct an isotopy invariant of a link, which requires $\mathrm{tr}(\psi_n(b))$ to be invariant under the Markov moves:

$$\mathrm{tr}(\psi_n(ab)) = \mathrm{tr}(\psi_n(ba)),$$
$$\mathrm{tr}(\psi_n(b)) = \mathrm{tr}(\psi_{n+1}(b\sigma_n^{\pm1}))$$

However, this fails, since we require $\mathrm{tr}_2 R^{\pm1} = id_V$ but $\mathrm{tr}_2 R^{-1} \neq id_V$. By using $h = \begin{bmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2}\end{bmatrix}$, and using the properties $R\cdot(h\otimes h) = (h\otimes h)\cdot R$ and $\mathrm{tr}_2((id_V\otimes h)\cdot R^{\pm1}) = id_V$ obtained by construction of R, we obtain the invariant $tr(h^{\otimes n}\cdot\psi_n(b))$, since

$$\begin{aligned}\mathrm{tr}(h^{\otimes n}\cdot\psi_n(ab)) &= \mathrm{tr}(h^{\otimes n}\cdot\psi_n(a)\psi_n(b)) = \mathrm{tr}(\psi_n(b)\cdot h^{\otimes n}\cdot\psi_n(a)) \\ &= \mathrm{tr}(h^{\otimes n}\cdot\psi_n(b)\psi_n(a)) = \mathrm{tr}(h^{\otimes n}\cdot\psi_n(ba)) \text{ and}\end{aligned}$$

$$\begin{aligned}\mathrm{tr}(h^{\otimes(n+1)}\cdot\psi_{n+1}(\sigma_n^{\pm1}b)) &= \mathrm{tr}(h^{\otimes(n+1)}\cdot(id_V^{\otimes(n-1)}\otimes R^{\pm1})\cdot\psi_{n+1}(b)) \\ &= \mathrm{tr}(h^{\otimes n}\cdot\psi_n(b)).\end{aligned}$$

The skein relation of this invariant is the same as that of the Jones polynomial since

$$t^{-1}R - tR^{-1} = (t^{-1/2} - t^{1/2})id_{V\otimes V}$$

and R, R^{-1} and $id_{V\otimes V}$ correspond to the three types of crossings. Since the value of the unknot under this invariant is $t^{1/2}+t^{-1/2}$, its value is $t^{1/2}+t^{-1/2}$ times the Jones polynomial $V_L(t)$. This gives an alternate definition of the Jones polynomial in terms of a braid b and its closure L:

$$\mathrm{tr}(h^{\otimes n}\cdot\psi_n(b)) = (t^{1/2}+t^{-1/2})V_L(t)$$

References

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