

HITTING PROBABILITIES IN 3-PLAYER BETTING GAMES

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Background settings of 3-player betting games

This project is motivated by Professor Persi Diaconis' paper[3]. The focus of this study is on the hitting probabilities in 3-player betting games. The rules of the games can be seen as the 3-player gambler's ruin problem. We set the fortune for each player at n -th round is denoted as $(X_n, Y_n, Z_n) \in \mathbb{N}_0^3$, and set $(X_0, Y_0, Z_0) = (x, y, z)$. We assign player1 with X_n and as the protagonist when comes to winning or losing. The betting rule is \$1 bet. Other variation of betting rules such as all-in bets are not discussed in this paper. The all-in betting rule is discussed in the unpublished paper: "All in! Poker sequences" [1] worked by Professor Mark Holmes and Professor Omer Angel.

Winning in this game means $X_n = x + y + z$, for some $n \in \mathbb{N}$. For both games, once 1 player is eliminated, then it turns into 2-player gambler's ruin problem. The games will continue until only 1 player remains.

We investigate hitting probabilities under 2 different procedure settings, which will be consistently referred as:

- Game1:** 2 players are uniformly chosen to play at each round.
- Game2:** All 3 players play at each round. If 2 players happen to be eliminated at the same round, toss a fair coin to decide who is to be declared as being eliminated first.

Definition of Martingale

Definition. A filtration [2] is an increasing sequence of σ -fields, where n -th field denoted by \mathcal{F}_n . Now for the stochastic process $(X_n)_{n \geq 0}$, if $X_n \in \mathcal{F}_n$, then this process is said to be adapted to $(\mathcal{F}_n)_{n \geq 0}$. $(X_n)_{n \geq 0}$ is a Martingale if the following is satisfied:

- $\mathbb{E}(|X_n|) < \infty$, $n \geq 0$
- $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$, $n \geq 0$

If X_n denotes the fortune of player1 in a game at the n -th round, then notice the 2nd definition tells us player1's future fortune are expected to stay at player1's most recent fortune. The \mathcal{F}_n contains information about fortunes of all players up to the n -th round.

Martingale Property: If $(X_n)_{n \geq 0}$ is a Martingale, then $\mathbb{E}(X_n) = \mathbb{E}(X_0)$, $n \geq 0$.

Proof. Since $(X_n)_{n \geq 0}$ is a Martingale, then $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$, $n \geq 0$. Taking expectation on both side yields $E(X_{n+1}) = E(X_n)$ for $n \geq 0$. Then by induction we have the result $\mathbb{E}(X_n) = \mathbb{E}(X_0)$, $n \geq 0$. \square

Optional Stopping Theorem

Optional Stopping Theorem [2]: For the Martingale process $(X_n)_{n \geq 0}$. Let stopping time be T . Then we have:

$$\mathbb{E}(X_T) = \mathbb{E}(X_0)$$

If one of the following conditions hold:

- The stopping time T is bounded.
- $\mathbb{E}(T) < \infty$ and $\mathbb{E}(|X_{n+1} - X_n| | \mathcal{F}_n)$ is bounded.
- $|X_n|$ is bounded for all $n \geq 0$.

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References

- [1] Omer Angel and Mark Holmes. "All in!" poker sequences". In: (2022).
- [2] Konstantin Borovkov. *Elements of stochastic modelling*. World Scientific Publishing Company, 2014.
- [3] Persi Diaconis and Stewart N Ethier. "Gambler's Ruin and the ICM". In: *Statistical Science* 37.3 (2022), pp. 289–305.

Martingale: Winning probability in 3-player games

Now let's derive the winning probability of player1 in 3-player fair betting game. We have $(X_0, Y_0, Z_0) = (x, y, z)$. For $(X_n)_{n \geq 0}$ over the state space $S = \{0, 1, \dots, x + y + z\}$. The stopping time is denoted as $T := \inf\{n : X_n \in \{0, x + y + z\}, n \geq 0\}$. $W := \{X_T = x + y + z\}$ The distribution of X_T is:

$$X_T = \begin{cases} 0, & \text{w.p. } \mathbb{P}(W^c) \\ x + y + z, & \text{w.p. } \mathbb{P}(W) \end{cases}$$

Notice in both games, once 1 of the players being eliminated, the dynamic of this process changes and becomes the same Martingale in the 2-player gambler's ruin problem. Also, in game2 we can have 2 players being eliminated at the same round. So we define $\tau := \inf\{n : (\mathbb{1}_{\{X_n=0\}} + \mathbb{1}_{\{Y_n=0\}} + \mathbb{1}_{\{Z_n=0\}}) \in \{1, 2\}, n \geq 0\}$ as the first time of the dynamic change. Also, whilst 3 players are all in this game, we need the sequence of (X_n, Y_n) to determine if there has been an elimination. Recap on **Game1**: 2 players are uniformly chosen to play at each round.

Proof. We first prove $(X_n)_{n \geq 0}$ is a Martingale.

- $\mathbb{E}(|X_n|) < \sup\{S\} = x + y + z < \infty$
- $\mathbb{E}(X_{n+1} | \mathcal{F}_n)$
 $= \mathbb{E}(X_{n+1} | (X_0, Y_0), \dots, (X_n, Y_n)) \mathbb{1}_{\{\tau \leq n\}}$
 $+ \mathbb{E}(X_{n+1} | (X_0, Y_0), \dots, (X_n, Y_n)) \mathbb{1}_{\{\tau > n\}}$
 $= [X_n \cdot \mathbb{1}_{\{X_n \in \{0, x+y+z\}\}} + \frac{1}{2}(X_n + 1) + \frac{1}{2}(X_n - 1)] \cdot \mathbb{1}_{\{0 < X_n < x+y+z\}} \mathbb{1}_{\{\tau \leq n\}}$
 $+ [\frac{1}{3}(X_n + 1) + \frac{1}{3}(X_n - 1) + \frac{1}{3}(X_n)] \cdot \mathbb{1}_{\{\tau > n\}}$
 $= [X_n \cdot \mathbb{1}_{\{X_n \in \{0, x+y+z\}\}} + X_n \cdot \mathbb{1}_{\{0 < X_n < x+y+z\}}] \cdot \mathbb{1}_{\{\tau \leq n\}} + X_n \cdot \mathbb{1}_{\{\tau > n\}}$
 $= X_n$

\square

Recap on **Game2**: All 3 players play at each round. The proof logic is similar to the proof for game1.

For **game1** and **game2**, both processes are Martingale. Also, this process is an unbiased random walk on a finite state space, which means $\mathbb{E}(T) < \infty$ and $\mathbb{E}(|X_{n+1} - X_n| | \mathcal{F}_n) \leq 2$. Therefore, by **Optional Stopping Theorem**:

$$\begin{aligned} \mathbb{E}(X_T) = \mathbb{E}(X_0) &\implies 0 \cdot \mathbb{P}(W^c) + (x + y + z) \cdot \mathbb{P}(W) = x \\ &\implies \mathbb{P}(W) = \frac{x}{x+y+z} \end{aligned}$$

In conclusion, the winning probability for player1 is $\frac{x}{x+y+z}$ in both 3-player betting games, which is just the ratio of initial fortune and total fortune.

Losing probability in 3-player betting game

Now we consider the losing probability in 3-player betting games with elimination orders. Notice that player1 can be eliminated first or second. In specific we are only interested in the event of player1 being eliminated first, and we denote such event as **L**. **Notice: We will refer the probability of event L as losing probability for convenience.** Instead of deriving the general analytical solution for it, our focus is on the behaviour of its probability function, parameterised by initial fortune of player1.

The motivation here originates from the Texas hold'em poker tournament. Imaging the final table is left with last 3 players, and player1 wants to analyse the probability. The exact probability is to hard to calculate, so player1 wants to know if there are any ways to decrease the losing probability by manipulating fortunes under rules. For example, player1 decides to risk \$1 by intentionally losing to player2 who has low fortune, and hoping for that the increase in player2's fortune will increase player3's loss rate more than player1's loss rate, and will not threat player1 too much. If it is true, then this decreases player1's losing probability, as player2 is used to weaken player3 on purpose. Therefore, we want to investigate if there is a chance to benefit from this kind of betting strategy, though our intuition clearly says more fortune means lower losing probability.

Monotonic decreasing proof for losing probability

Proposition. Let $f(x) := P_{(x,y,z)}$, where $y, z \in \mathbb{N}$ are fixed. We claim $f : \mathbb{N} \rightarrow \mathbb{R}$ is monotone decreasing.

Game1:

Proof. We want to show $f(x+1) < f(x)$ for monotone decreasing. We construct the outcome of each round via iid $U_i \sim U(0, 1)$, $i \geq 1$. Let $\eta(U_i)$ be the outcome function for each round.

$$\eta(U_i) = \begin{cases} (1, -1, 0), & U_i \leq \frac{1}{3}\alpha \\ (1, 0, -1), & \frac{1}{3}\alpha < U_i \leq \frac{2}{3}\alpha \\ (0, -1, 1), & \frac{2}{3}\alpha < U_i \leq \frac{2}{3}\alpha + \frac{1}{6} \\ (0, 1, -1), & \frac{2}{3}\alpha + \frac{1}{6} < U_i \leq \frac{2}{3}\alpha + \frac{1}{3} \\ (-1, 1, 0), & \frac{2}{3}\alpha + \frac{1}{3} < U_i \leq \frac{1}{3}\alpha + \frac{2}{3} \\ (-1, 0, 1), & \frac{1}{3}\alpha + \frac{2}{3} < U_i \leq 1 \end{cases}$$

We construct 2 processes of game1 based on the same set of $\eta(U_i)$, $(X_n, Y_n, Z_n)_{n \geq 0}$ and $(X'_n, Y_n, Z_n)_{n \geq 0}$. We denote $T^{(i)} := \inf\{n : X_n^{(i)} \in \{0, x + y + z\}, n \geq 0\}$. Under such construction, the two processes of players' fortune at the n -th round given $X_0 = x$ and $X'_0 = x + 1$ are:

$$(X_n, Y_n, Z_n) = (x, y, z) + \sum_{i=1}^{n \wedge T'} \eta(U_i) \quad (1)$$

$$(X'_n, Y_n, Z_n) = (x + 1, y, z) + \sum_{i=1}^{n \wedge T'} \eta(U_i) \quad (2)$$

Notice we use the same set of $\{U_i, i > 1\}$ to construct 2 processes, which means we force 2 trajectories to share the same movement from the beginning. Now if $(X'_n, Y_n, Z_n)_1 = X'_n$ reaches 0 first among players at the n -th round (i.e. $n = T'$). Then this event $A' = \{X'_{T'} = 0, Y_i \neq 0, Z_i \neq 0, \forall i \leq T' | T' \in \mathbb{N}\}$ implies:

$$x + 1 + \left[\sum_{i=1}^{T'} \eta(U_i) \right]_1 = 0 \implies \left[\sum_{i=1}^{T'} \eta(U_i) \right]_1 = -(x + 1)$$

Therefore, there exists:

$$T_{set} = \{t_j < T' : \left[\sum_{i=1}^{t_j} \eta(U_i) \right]_1 = -x, j \in \mathbb{N}\} \implies \inf\{T_{set}\} = T$$

This means a set of all t_j such that $X_{t_j} = 0$. Taking the infimum then simply yields our stopping time T for the first process. Let us denote this loss event of player1 in the first process as $A = \{X_T = 0, Y_i \neq 0, Z_i \neq 0, \forall i \leq T | T \in \mathbb{N}\}$.

Now since T_{set} is bounded above by T' , we have $\inf\{T_{set}\} = T < T' = n$. This proves if player1 is eliminated first in the second process $(X'_n, Y_n, Z_n)_{n \geq 0}$ at n -th round, then it implies player1 will be eliminated in the first process $(X_n, Y_n, Z_n)_{n \geq 0}$ before n -th round for sure.

So we have: $A' \subset A$. Therefore:

$$f(x+1) = P(A') < P(A) = f(x)$$

Hence, we prove the losing probability of player1 is monotone decreasing in game1. \square

Game2: The proof follows in a similar way as in the proof of game1. It is just a matter of changing the $\eta(U_i)$ according to game2's distribution.

In conclusion, the sacrificing betting strategy does not yield benefit for our player. However, in a paper[1] worked by Professor Mark Holmes and Professor Omer Angel, it has already been proved that with all-in betting rule, such betting strategy does yield benefit for our player.

Example of explicit calculation of losing probability of game1 for small state space

Consider the total fortune of our game is $N \in \mathbb{N}$. Then the 3-player gambler's ruin problem of **game1** can be seen as a random walk over the state space $S_N := \{(X_n, Y_n, Z_n) \in \mathbb{N}_0^3 : X_n + Y_n + Z_n = N | n \geq 0\}$ [3]. We will be looking at S_4 , since losing probability over S_3 is just uniform.

Again, let $P_{(x,y,z)}$ be the losing probability of player1, given $(X_0, Y_0, Z_0) = (x, y, z)$. Here, we we will be explicitly computing $P_{x,y,z}$ over S_4 . Now over S_4 , $x \in \{1, 2\}$, so we will be interesting in $P_i, \forall i \in I := \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}$.

Let $J := \{(0, 1, 3), (0, 3, 1), (0, 2, 2)\}$, where J is the set of states that player1 loses first. We define $p_{i,J} := \mathbb{P}(X_n \in J \text{ for some } n \geq 1 | X_0 = i)$, where $i \in S_4$.

The dynamics of this game is simple from the diagram. The probability assigned on each arrow path is uniform.

Now back to $p_{i,J}, i \in S_4$. We are only interested in the probability of reaching J , which is the boundary at the bottom and we are not interested in who is the final winner of the game. Therefore, we assume the states in J to be absorbing which aligns with player1's point view of being ruined. Also notice that those no comeback states yield 0 probability for $p_{i,J}$.

We first calculate $P_{(1,1,2)}$:

$$P_{(1,1,2)} = p_{(1,1,2)J}$$

Also, we know by symmetric:

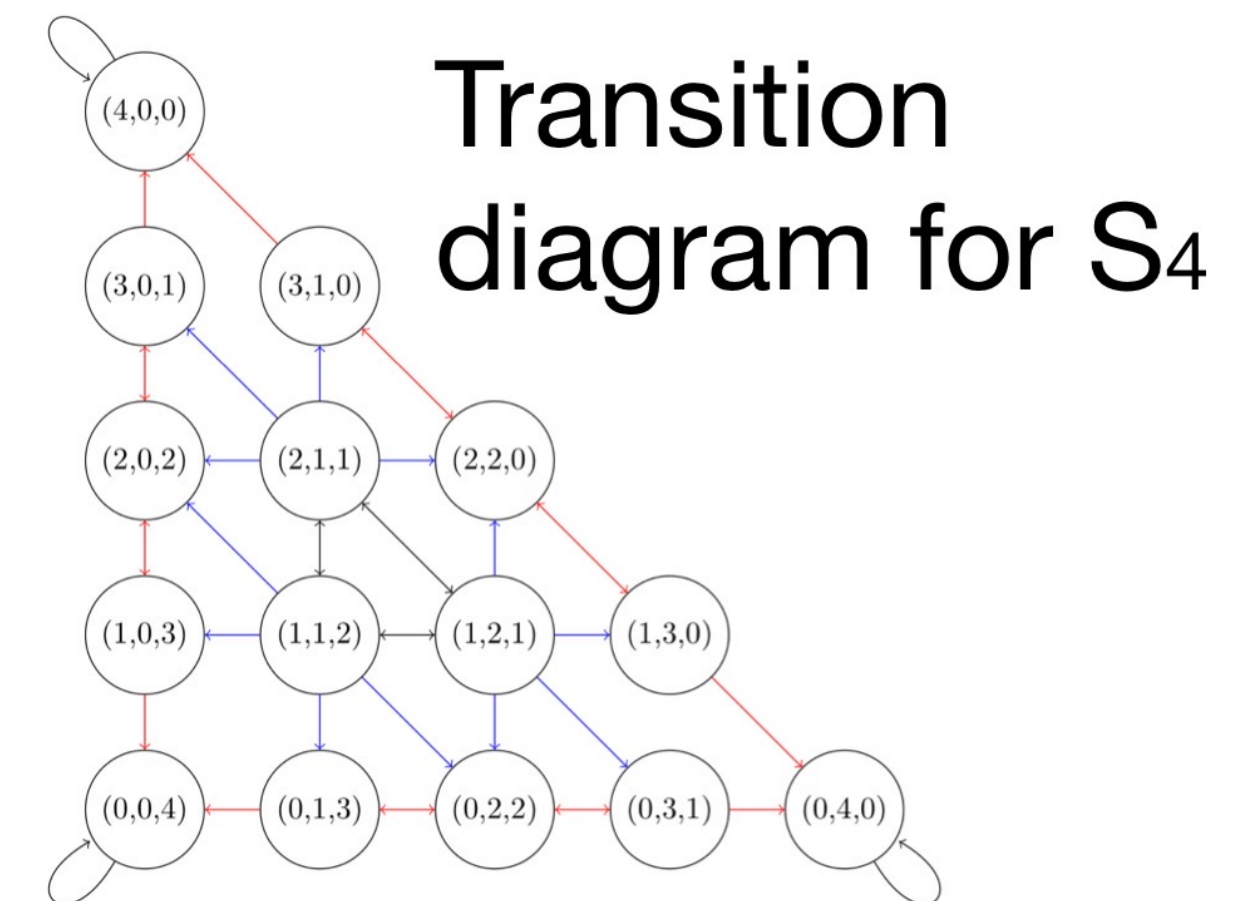
$$P_{(1,1,2)} = P_{(1,2,1)J} = p_{(1,1,2)J} = p_{(1,2,1)J} \quad (3)$$

$$\begin{aligned} P_{(1,1,2)J} &= \frac{1}{6}(p_{(0,1,3)J} + p_{(1,0,3)J} + p_{(2,0,2)J} \\ &\quad + p_{(2,1,1)J} + p_{(1,2,1)J} + p_{(0,2,2)J}) \\ &= \frac{1}{6}(2 + P_{(2,1,1)J} + p_{(1,2,1)J}) \end{aligned} \quad (4)$$

$$\begin{aligned} P_{(2,1,1)J} &= \frac{1}{6}(p_{(2,0,2)J} + p_{(3,0,1)J} + p_{(3,1,0)J} \\ &\quad + p_{(2,2,0)J} + p_{(1,1,2)J} + p_{(1,2,1)J}) \\ &= \frac{1}{6}(P_{(1,1,2)J} + p_{(1,2,1)J}) \end{aligned} \quad (5)$$

From (3)(4)(5), we can solve $p_{(1,1,2)J} = \frac{3}{7}$.

So we have $P_{(1,1,2)} = P_{(1,2,1)} = \frac{3}{7}$. This then leaves $P_{(2,1,1)} = \frac{1}{7}$. Therefore, in game1, we can conclude that over S_4 , the probability of player1 losing first given $X_0 = 1$ is $\frac{3}{7}$; and given $X_0 = 2$ is $\frac{1}{7}$.



Transition diagram for S_4