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Introduction

In category theory, there are two types of monoidal (tensor) categories: the 'strict' and the 'relaxed'. The difference between them is whether the 'strict' equality sign is relaxed or not in different 'coherence conditions'. 'Associators', the ones that satisfy one of the relaxed coherence conditions (pentagon identity), play important roles in classifying the relaxed monoidal category. Therefore, the main purpose of this poster is to study a way to classify associators in a rather special type monoidal category, that is constructed from a finite abelian group G, with one 'non-invertible' object N, which behaves 'strangely' under the defined tensor product:

- \blacktriangleright g \otimes h = gh
- $\blacktriangleright \mathcal{N} \otimes g = \mathcal{N} = g \otimes \mathcal{N}$
- $\blacktriangleright \mathcal{N} \otimes \mathcal{N} = \bigoplus_{g \in G \setminus \{\mathcal{N}\}} g$

Monoidal Category

A **monoidal category** is usually denoted as $(C, \otimes, E, \alpha, \lambda, \rho)$ [2], the following explains each components:

A category [2] is just a directed graph with vertices being objects set (*O*) and directed edges being the set of morphisms(\mathcal{A}) between objects with composition rules between arrow and identity function id : $O \rightarrow \mathcal{A}, A \mapsto id_A$, where $id_A \in Hom(A, A)$, that then needs to satisfy two axioms: Associativity of composition and existence of left and right identity morphism for a morphism. A **Tensor product** [4] could be intuitively thought of as a defined process of 'fusion' between two mathematical objects.

Coherence Condition:

Associativity: 'strict' tensor categories require $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, while 'relaxed' ones relax to require the two above equal up to a natural isomorphism, which we call an 'associator':

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

the coherence condition that all associators need to satisfy is called the pentagon equation or pentagon diagram:[2]



Unit law: Any tensor category needs to have a 'tensor unit', we define as *E* and satisfaction of this axiom gives rise to two natural isomorphisms, left and right unitors λ, ρ :

$$\lambda_A: E \otimes A \cong A, \ \rho_A: A \otimes E \cong A$$

the coherence condition that each pair of unitors needs to satisfy collectively with any associator is called the triangular diagram:



Classification of Monoidal Category

Case I

We are looking at $\omega(g, h, l) \in \text{Hom}((g \otimes h) \otimes l, g \otimes (h \otimes l)) = \mathbb{K}^{\times}$ with pentagon equation below:

$$\omega(h, k, l) \circ \omega(g, hk, l) \circ \omega(g, h, k) = \omega(g, h, kl) \circ \omega(gh, k, l)$$

(1)

Case II

The tensor product result is consistent if we have on N in triplet regardless of the position of N, and we define a new class of associators: $\phi \in Hom(N, N) = \mathbb{K}^{\times}$ and more specifically, there are three types of associators under this class:

- $\bullet \quad \dot{\phi}^{(1)}(g,h) \in \operatorname{Hom}((g \otimes h) \otimes \mathcal{N}, g \otimes (h \otimes \mathcal{N}))$
- ▶ $\phi^{(2)}(g,h) \in \operatorname{Hom}((g \otimes N) \otimes h, g \otimes (N \otimes h))$
- $\phi^{(3)}(g,h) \in \operatorname{Hom}((\mathcal{N} \otimes g) \otimes h, \mathcal{N} \otimes (g \otimes h))$

Because the set of equations is highly symmetric, we will only display 2 diagrams



$$\phi^{(1)}(g,h)\phi^{(1)}(l,gh)\omega(l,g,h) = \phi^{(1)}(l,g)\phi^{(1)}(lg,h)$$
(2)
$$\phi^{(2)}(g,h)\phi^{(2)}(l,h) = \phi^{(2)}(lg,h)$$
(3)
$$\phi^{(2)}(l,h)\phi^{(2)}(l,g) = \phi^{(2)}(l,gh)$$
(4)

$$\varphi^{(1)}(l,h)\varphi^{(1)}(l,g) = \varphi^{(1)}(l,gh)$$
(4)

$$\omega(l,g,h)\phi^{(3)}(lg,h)\phi^{(3)}(l,g) = \phi^{(3)}(l,gh)\phi^{(3)}(g,h)$$
(5)

Case III:

Define $\psi \in \text{Hom}(\bigoplus_{g \in G \setminus \{N\}} g, \bigoplus_{h \in G \setminus \{N\}} h) = \mathbb{K}^{|G|}$, and one should notice ψ 's are diagonal matrices $\psi^{(1)}(g, l) \in \text{Hom}((g \otimes N) \otimes N, g \otimes (N \otimes N))$

- $\blacktriangleright \psi^{(2)}(g,l) \in \operatorname{Hom}((\mathcal{N} \otimes g) \otimes \mathcal{N}, \mathcal{N} \otimes (g \otimes \mathcal{N}))$
- ► $\psi^{(3)}(g, l) \in \operatorname{Hom}((N \otimes N) \otimes g, N \otimes (N \otimes g))$

I will only display 2 diagrams[1]



Vec_G

 \mathbf{Vec}_{G} , the category with objects being G-graded vector spaces and morphisms being grade-preserving linear maps, where G is a group. We denote each G-graded subspace as $S_g = \mathbb{K}$, and *G*-grade-preserving property of mapping allows Hom $(\mathcal{S}_g, \mathcal{S}_H) = \delta_{g,h} \mathbb{K}$, where δ is the Kronecker delta function δ .

As a result, by the pentagon equation in the last section and defining a projection isomorphism mapping id $\otimes \omega \mapsto \omega$ and $\omega \otimes id \mapsto \omega$, the pentagon equation is

 $\omega(g, h, kl) \circ \omega(gh, k, l) = \omega(h, k, l) \circ \omega(g, hk, l) \circ \omega(g, h, k)$

Moreover, if we have ω_1 and ω_2 both satisfy the pentagon equation, we can prove that their product is commutative and also satisfies the pentagon equation

Moreover, the coherence condition resembles something called 'coboundary' condition in group cohomology, therefore, We now introduce the set of 3-cocycles, $Z^3(G, \mathbb{K}^{\times})[3]$, which is indeed the set of all functions satisfy such condition, very much mimics the pentagon equation: Based on the fact that \mathbb{K} is a field and the commutative property of the product of associators, we can assert that $Z^3(G, \mathbb{K}^{\times})$ is an Abelian group.

Next, we will focus on a subset of Z^3 that is constructed from some arbitrary function $f: G \times G \to \mathbb{K}^{\times}$, such that $\omega: G \times G \times G \to \mathbb{K}^{\times}$, $\omega(g, h, k) = \frac{f(h,k)f(g,hk)}{f(g,h)}$, the set of these ω 's is the same as subset $B^3(G, \mathbb{K}^{\times})$ of Z^3 , called the set of 3-coboundaries. As a subset of Abelian group Z^3 , it is abelian and a normal subgroup of Z^3 , which implies the existence of quotient group Z^3/B^3 Equivalence Classes and Basis changes in Hom($S_g \otimes S_h, S_{gh}$)

it is not hard to discover that equivalence classes in quotient group Z^3/B^3 govern the relationship between $\mathcal{B}_{2}^{(gh,k)}$ and $\mathcal{B}_{2}^{(g,hk)}$ if we dig deep into the subtly from $\mathcal{S}_{(gh)k}$ and $\mathcal{S}_{g(hk)}$ And, we find out that $H^3(G,\mathbb{K}) := Z^3(G,\mathbb{K}^{\times})/B^3(G,\mathbb{K}^{\times})$ called the third integral group cohomology of group G and $(\mathbb{K}^{\times}, \times)$ in group cohomology.

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Reference

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$$\psi^{(1)}(h, g^{-1}l)\psi^{(1)}(g, l)\phi^{(1)}(g, h) = \omega(g, h, h^{-1}g^{-1}l)\psi^{(1)}(gh, l)$$

$$\psi^{(2)}(h, g^{-1}l)\phi^{(2)}(g, h) = \psi^{(2)}(h, l)$$

$$\phi^{(1)}(g, h)\psi^{(2)}(gh, l)\phi^{(3)}(g, h) = \psi^{(2)}(g, l)\psi^{(2)}(h, l)$$
(8)

 $\phi^{(1)}(g,h)\psi^{(2)}(gh,l)\phi^{(3)}(g,h) = \psi^{(2)}(g,l)\psi^{(2)}(h,l)$

$$\psi^{(3)}(h, g^{-1}l)\omega(g, g^{-1}lh^{-1}, h)\psi^{(1)}(g, lh^{-1}) = \psi^{(1)}(g, l)\psi^{(3)}(h, l)$$
(9)

$$\phi^{(2)}(g,h)\psi^{(2)}(g,lh^{-1}) = \psi^{(2)}(g,l)$$
(10)

$$\phi^{(3)}(g,h)\psi^{(3)}(h,l)\psi^{(3)}(g,lh^{-1}) = \psi^{(3)}(gh,l)\omega(lh^{-1}g^{-1},g,h)$$
(11)

Case IV: Due to various considerations, we define a new class of associators $\alpha = \sum_{h \in G \setminus \{N\}} \beta(h, k) \in \text{Hom}((N \otimes N) \otimes N, N \otimes (N \otimes N))$ where h is in the source triplet and k is the choice made in the target triplet, which is fixed. Then

$$\psi^{(3)}(k^{-1}g,g)\phi^{(3)}(k,g)\beta(h,k^{-1}g) = \beta(h,k)\phi^{(2)}(h,g)$$
(12)

$$\psi^{(2)}(g,k)\beta(hg,k)\psi^{(3)}(g,hg) = \beta(h,k)\phi^{(1)}(h,g)$$
(13)

$$\psi^{(1)}(g,kg)\beta(h,kg)\psi^{(2)}(g,h) = \phi^{(3)}(g,k)\beta(h,k)$$
(14)

$$\beta(g^{-1}h,k)\phi^{(1)}(g,g^{-1}h)\psi^{(1)}(g,h) = \phi^{(2)}(g,k)\beta(h,k)$$
(15)

Case V: This case has 4 N involved in the pentagon equation.



By choosing suitable trivial basis changes, we can simplify the systems of equations, and solve to get:

 $\sum_{h'} \beta(h',k)\psi^{(2)}(h',l)\beta(h,h') = \delta_{h^{-1}l,k}\psi^{(3)}(h^{-1}l,l)\psi^{(1)}(h,l)$

- $\blacktriangleright \omega = 1, \phi^1 = \psi^3 = 1, \phi^3 = \psi^1 = 1$
- $\phi^2 = \psi^2 = (\mathcal{F} : G \times G \to \mathbb{K}^{\times})$
- $\beta(e, e)^2 \sum_{h'} \phi^{(2)}(h', h^{-1}lk^{-1}) = \delta_{h^{-1}l,k} \text{ and } \beta(e, e)^2 \sum_{h'} \phi^{(2)}(h', e) = 1$ $\Rightarrow \beta(e, e)^2 |G| = 1 \Rightarrow \beta(h, k) = \frac{\gamma}{\mathscr{F}}, \text{ where } \gamma^2 = |G|^{-1}$