Colourings of oriented abelian Cayley graphs

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Introduction and definitions

A **graph homomorphism** $G \to H$ is a mapping h from the vertices of G to the vertices of H such that if uv is an edge of G, h(u)h(v) is an edge of H. When H is the complete graph K_t, **homomorphisms to H correspond to proper t-colourings** of G.

Using homomorphisms, we can **extend the definition of proper colouring to directed graphs** in a way that respects arc direction.

Recall an **oriented graph** $G = (V, A)$ is a directed graph constructed by orienting the edges of an underlying simple graph.

An **oriented k-colouring of** G is defined as a homomorphism ϕ: $G \to T_k$ where T_k is a tournament on k vertices. This definition sets two conditions for a valid oriented colouring:

1. $uv \in A(G) \Rightarrow \phi(u) \neq \phi(v)$
2. $uv, xy \in A(G), \phi(u) = \phi(y) \Rightarrow \phi(v) \neq \phi(x)$

The first condition requires **adjacent vertices receive different colours**. The second condition additionally requires that **all arcs from a given colour class to another must have the same direction**. The direction of an arc ij in T_k gives us the direction of arcs between colours i and j.

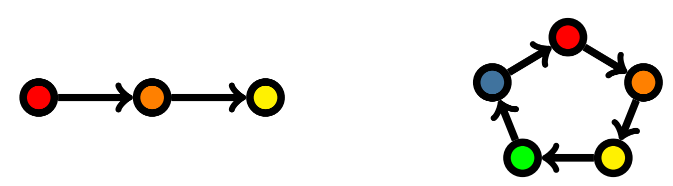

Figure 1. Vertices connected by a 2-dipath require different colours, and the directed 5-cycle requires five colours.

The **oriented chromatic number** χ_o**(G) of a simple (undirected) graph** G is the smallest k such that all possible orientations of G admit an oriented k-colouring. If $\mathcal{F}$ is a family of graphs, $\chi_o(\mathcal{F})$ is the smallest k such that all orientations of all graphs in $\mathcal{F}$ are k-colourable.

Let $\mathcal{F}$ be the family of orientations of connected cubic graphs. A long-standing conjecture[3] asserts $\chi_o(\mathcal{F}) = 7$.

Recall the **Cayley graph** $\mathbf{\Gamma = C(Gp, S)}$ is constructed from a group Gp, such that:

- each element g of Gp is represented by a vertex g in Γ;
- S is an inverse-closed subset of the elements of Gp (not necessarily a generating set for Gp); and
- there is an edge (g, gs) for every $g \in V(\Gamma)$, $s \in S$.

In this project we show all orientations of cubic abelian Cayley graphs with no source or sink vertex are 7-colourable. Moreover, every such orientation admits a homomorphism to the Paley tournament on 7 vertices.

What are the cubic abelian Cayley graphs?

Recall Y_n is the prism graph formed by the skeleton of an n-prism. For even n, let G_n be the circulant graph $C(\mathbb{Z}_n, \{\pm 1, n/2\})$.

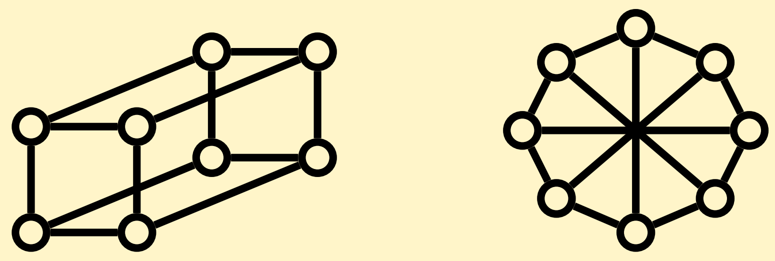

Figure 2. Y_4 (left) and G_8 (right)

Theorem:
If Γ is a cubic Cayley graph on an abelian group, then there exists $t \in \mathbb{N}$ such that every component of Γ is isomorphic to Y_t or G_t.

As Γ is undirected, loopless and cubic, it follows that the group identity $e \notin S$, order(Gp) is even and S has 3 distinct elements α, β and δ. As S is inverse-closed with odd cardinality, at least one element is self-inverse. Let $\beta = \beta^{-1}$. Let t be the order of α in Gp. We proceed in cases based on whether α is self-inverse.

Case 1: $\alpha = \alpha^{-1}$
Then as S is inverse-closed, every element of S is self-inverse. As Gp is abelian, multiplying an arbitary element u with any two elements of S induces the (undirected) 4-cycle, such as u,$u\alpha$,$u\alpha\beta$,$u\beta$. Multiplying each of these elements with δ will yield another copy of this 4-cycle, **yielding a copy of Y_4 as a component of Γ.**

Case 2: $\alpha \neq \alpha^{-1}$
Then $S = \{\alpha, \alpha^{-1}, \beta\}$ as S is inverse-closed. Through repeated multiplication starting with an arbitrary element u, α (together with α^{-1}) induces an undirected t-cycle: u,$u\alpha$,$u\alpha^2$,...,$u\alpha^{t-1}$.

If $\beta = \alpha^k$ for some k, the resulting component is as in Figure 3:

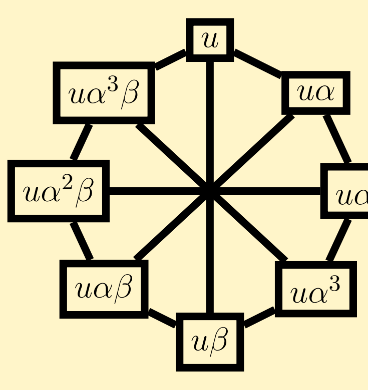

Figure 3. G_t as a component of Γ, for $t = 8$, $\beta = \alpha^4$

Otherwise if $\beta \neq \alpha^k$ for any k, the component is as in Figure 4:

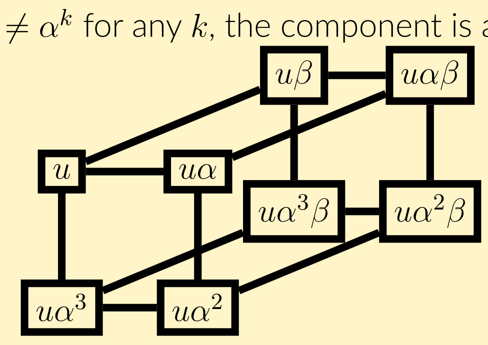

Figure 4. Y_t as a component of Γ, for $t = 4$

As u is arbitrary, the components of Γ are pairwise isomorphic. Hence for some $t \in \mathbb{N}$, Γ is either the disjoint union of copies of G_t or the disjoint union of copies of Y_t.

Relevant past results in oriented colouring

Oriented graph colouring has been well-studied:

- All oriented connected cubic graphs can be coloured with 8 colours[1].
- All orientations of ladder graphs (aka the grid $Gd(2, n)$) are 6-colourable for all n [2].

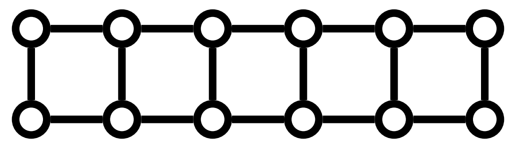

Figure 5. The ladder graph $Gd(2, 6)$

- We observe that components of Γ can be regarded as ladder graphs with the addition of two edges between the end vertices of the ladder.

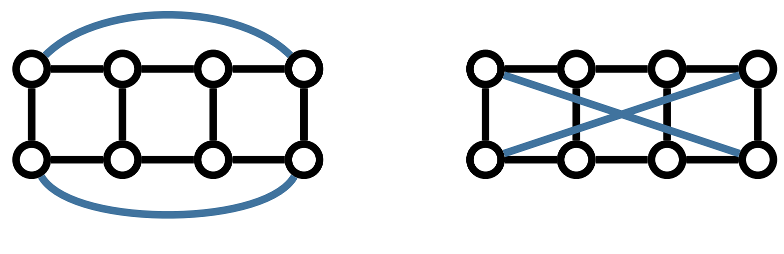

Figure 6. Y_4 (left) and G_8 (right)

- Past oriented colouring results frequently utilise homomorphisms to a Paley tournament QR_q, constructed from a prime power q, with vertices $\{0, 1, ..., q-1\}$ such that ij is an arc exactly when $j - i \not\equiv 0 \pmod{q}$ is a non-zero quadratic residue.
- Importantly, QR_q is an arc-transitive graph, meaning we can always fix a selected arc with a particular colouring.

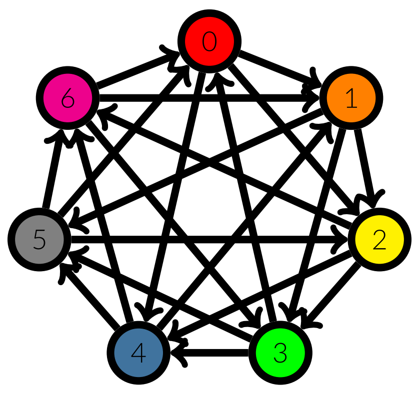

Figure 7. QR_7

Results for cubic abelian Cayley graphs

- $\chi_o \geq 7$.

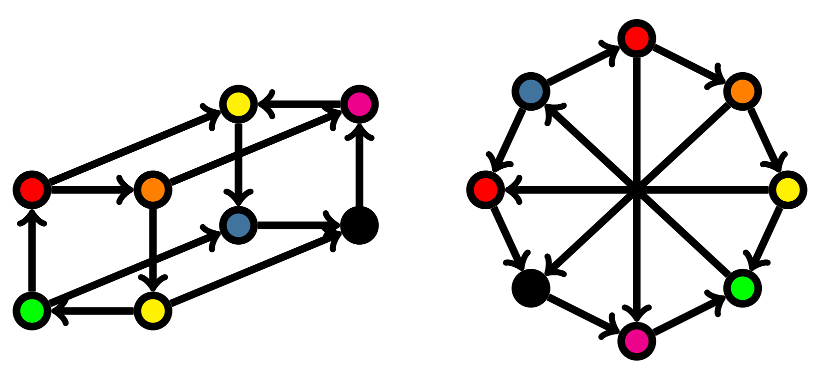

Figure 8. Orientations of Y_4 and G_8 which requires seven colours

- Not every orientation of a cubic abelian Cayley graph admits a homomorphism to QR_7. In all identified orientations where this homomorphism does not exist, we note adjacent source (3 outgoing arcs) and sink (3 incoming arcs) vertices.

Tiling cubic abelian Cayley graphs

- Let $\mathcal{C}$ be the family of orientations of cubic Cayley graphs on abelian groups with no sources or sinks. We find a homomorphism of $\Gamma \in \mathcal{C}$ to QR_7 by tiling.

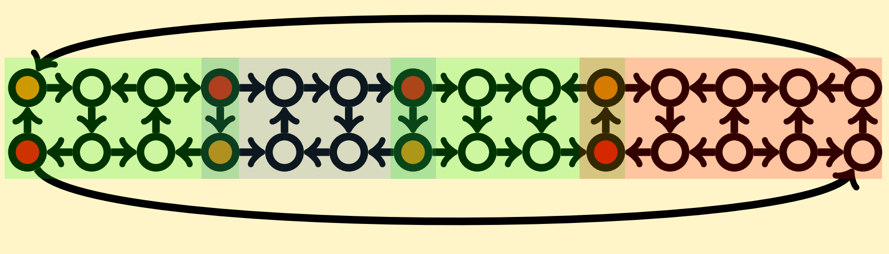

Figure 9. Tiling a random orientation of a component of Γ

- The first two required tiles, which we call **3-tiles**, are ladder graphs on 8 vertices. In the green, the end arcs are oriented differently, and in the blue, they are oriented the same.
- Over all possible orientations there are 2^9 tiles. **To ensure we can combine these tiles, we colour them so the first and last pairs of vertices use the same colours.**

Figure 10. The 3-tiles with partially fixed colouring

- By exhaustive search, all orientations of the blue 3-tile admit a homomorphism to QR_7, and all orientations of the green 3-tile without a source or sink admit a homomorphism to QR_7.
- **As QR_7 is arc-transitive, utilising and reversing vertex identification we can fix both end arcs with the same colours, as required to tile in a partially-overlapping way.**

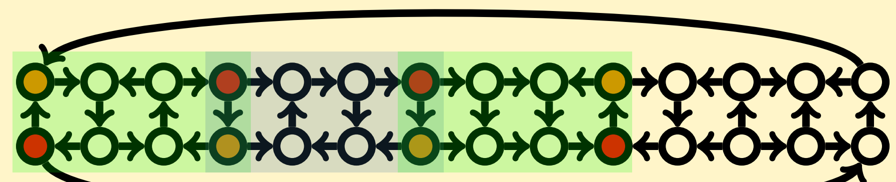

Figure 11. Tiling our 7-colouring until four tiles remain.

- We continue this tiling until we have 2, 3 or 4 remaining tiles (equivalently 4, 6 or 8 vertices uncoloured).

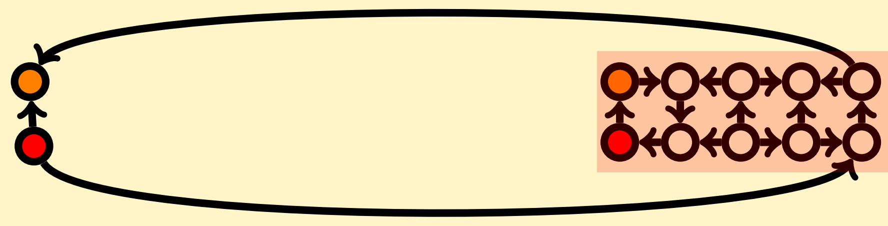

Figure 12. Remaining uncoloured vertices.

- **These cases are equivalent to requiring valid 3-, 4- and 5-tiles.** By exhaustive search, all source- and sink-free orientations of 4- and 5-tiles admit a homomorphism to QR_7 and we can complete the tiling.

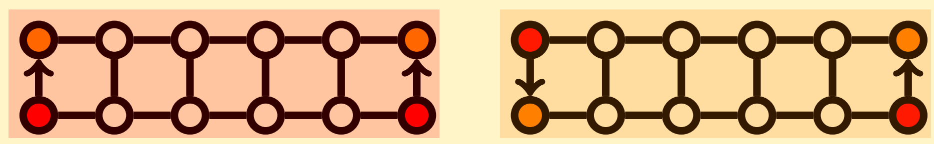

Figure 13. Over all possible orientations, there are 2^{15} 5-tiles

- Above in Figure 7 is a tiling of a Y_n component. The same tiles may be used to tile a G_n component.

As each tile admits a homomorphism to QR_7, when we compose the tiling we construct a homomorphism to QR_7. Doing this for each component constructs a homomorphism from Γ to QR_7.

Future directions for research

We propose two possible directions to extend this result to all orientations of cubic abelian Cayley graphs.

Both 5-tiles always admit a homomorphism to QR_7**, regardless of the presence of sources and sinks.**

- If this also holds for 6-, 7-, 8- and 9-tiles, the resulting tiling will extend the 7-colouring result as desired.
- However, this is very computationally intensive to check, with the 9-tile requiring up to approximately 2^{26} orientations to be assessed for homomorphism.

We generated early evidence that orientations of the ladder graph with one additional edge between end vertices are 6-colourable.

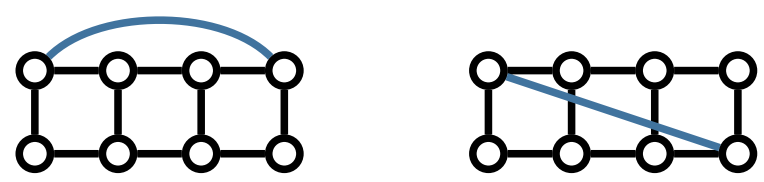

Figure 14. Equivalently, Γ minus an arbitrary edge uv

If this can be established rigorously, it may be possible to extend 6-colourings of orientations of these graphs to 7-colourings of orientations of Γ.

Acknowledgements and references

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