Topic 2: COMPLEX NUMBERS

To compliment our knowledge of functions of real variables, we now introduce basic properties and applications of complex numbers.

While originally a revolutionary concept, complex numbers are now used extensively in physics and engineering, in areas such as electromagnetic waves and electric circuits, and together with calculus form the mathematical study of complex analysis.

2.1 Introduction to Complex Numbers

We know that many polynomial equations have real solutions, for example:

$$\begin{aligned} x^2 - 1 &= 0 \\ \Rightarrow \qquad x^2 &= 1 \\ \Rightarrow \qquad x &= \pm 1 \end{aligned}$$

However, there are also many polynomial equations which do not have real solutions, for example:

$$\begin{aligned} x^2 + 1 &= 0 \\ \Rightarrow \qquad x^2 &= -1 \qquad (1) \end{aligned}$$

Since no real number can be squared to give -1, this equation has no real solutions for x.

However, if we define an 'imaginary number', denoted by i, such that

$$i^2 = -1$$

then the solutions to (1) become

$$\begin{aligned} x &= \pm\sqrt{-1} \\ &= \pm\sqrt{i^2} \\ &= \pm i \end{aligned}$$

So this imaginary number i can allow us to solve equations that we could previously not solve over the reals!

Definition:
A **complex number** (generally denoted z) is defined as a quantity consisting of a real number added to a multiple of the imaginary unit i. That is:

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

x is called the **real part** of z and is denoted $\text{Re}(z)$.

Note: y not iy

y is called the **imaginary part** of z and is denoted $\text{Im}(z)$.

The set of all complex numbers is denoted $\mathbb{C}$.

A complex number written in the form $z = x + iy$ is said to be in **cartesian form**.

Note that the reals $\mathbb{R}$ are a subset of the complex numbers $\mathbb{C}$. Why?

Any real number $x \in \mathbb{R}$ can be written $x + 0i \in \mathbb{C}$

Examples:

- Using the imaginary number i, write down an expression for $\sqrt{-25}$.

$$\sqrt{-25} = \sqrt{-1}\sqrt{25} = i5$$

- Simplify i^7.

$$i^7 = i^2 i^2 i^2 i = (-1)(-1)(-1)i = -i$$

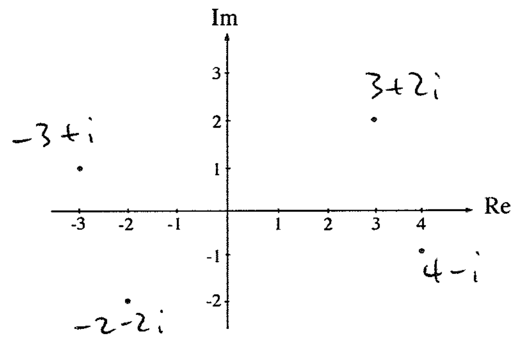
- For the complex number $z = 2 - 3i$, write down:

(i) $\text{Re}(z) = 2$
(ii) $\text{Im}(z) = -3$
(iii) $\text{Re}(z) - \text{Im}(z) = 2 - (-3) = 5$

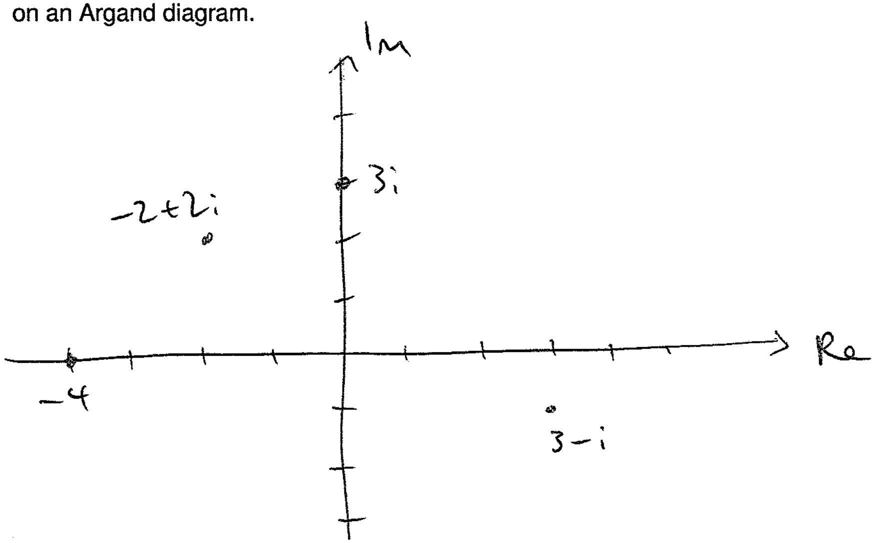
Argand Diagrams

Complex numbers can be represented graphically on an **Argand diagram**. We regard the complex number $z = x + iy$ as corresponding to a point on the xy-plane, where the x-axis is now called the **real axis** and corresponds to the real part of z, while the y-axis is the **imaginary axis** and corresponds to the imaginary part of z.

An Argand diagram is also called the **complex plane**.



Example: Sketch the complex numbers $3-i$, $-2+2i$, -4, and $3i$ on an Argand diagram.



2.2 Arithmetic of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers.

Equality: The complex numbers z_1 and z_2 are equal if and only if

$$\text{Re}(z_1) = \text{Re}(z_2) \quad \text{and} \quad \text{Im}(z_1) = \text{Im}(z_2)\,.$$

i.e. $a = c$ and $b = d$

Addition: We can add z_1 and z_2 as follows:

$$\begin{aligned} z_1 + z_2 &= (a+ib) + (c+id) \\ &= (a+c) + i(b+d) \end{aligned}$$

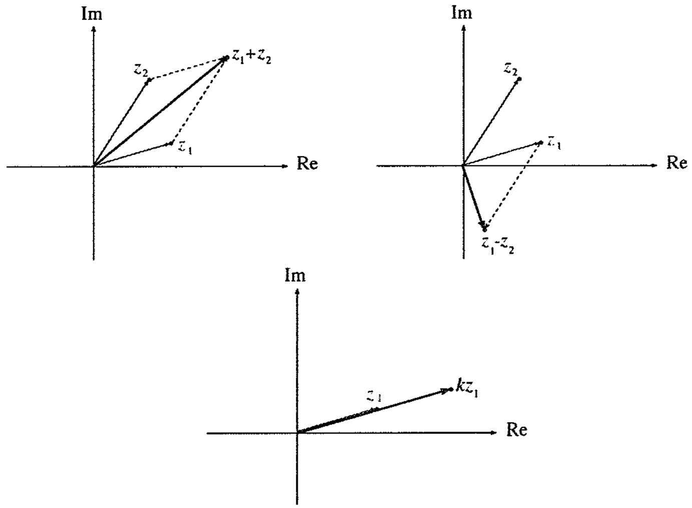
Subtraction: We can subtract z_1 and z_2 as follows:

$$\begin{aligned} z_1 - z_2 &= (a+ib) - (c+id) \\ &= (a-c) + i(b-d) \end{aligned}$$

Multiplication by $k \in \mathbb{R}$: We can multiply z_1 by $k \in \mathbb{R}$ as follows:

$$\begin{aligned} kz_1 &= k(a+ib) \\ &= (ka) + i(kb) \end{aligned}$$

Complex addition, subtraction, and multiplication by a real number, can be represented geometrically on an Argand diagram, and may remind you of vector arithmetic.



Multiplication of complex numbers

We can multiply two complex numbers by simply 'expanding the brackets', remembering that $i^2 = -1$.

$$\begin{aligned} z_1 z_2 &= (a+ib)(c+id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

The Complex Conjugate

In order to divide complex numbers, we first need to define the complex conjugate of a complex number.

> **Definition:** If $z = a + ib$ is a complex number, then the **complex conjugate** of z is denoted $\bar{z}$ ("z bar"), and is defined to be
> $$\bar{z} = a - ib\,.$$

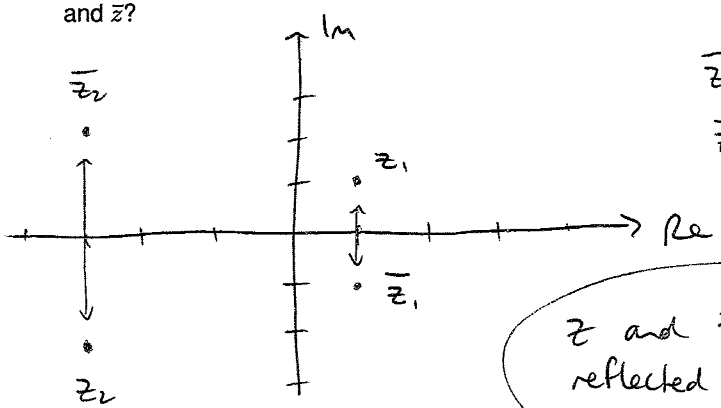
That is, the real part stays the same and the imaginary part changes sign.

Example: Write down the complex conjugate of:

(i) $-3 + 7i$ (ii) $2 - 5i$ (iii) $3i$ (iv) 4

$-3-7i$ $2+5i$ $-3i$ 4

Example: If $z_1 = 1 + i$ and $z_2 = -3 - 2i$, plot z_1, z_2, $\bar{z}_1$ and $\bar{z}_2$ on an Argand diagram. What is the graphical relationship between z and $\bar{z}$?



$\bar{z}_1 = 1 - i$

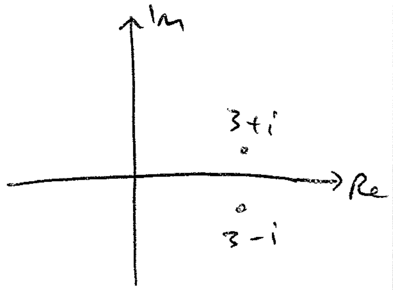
$\bar{z}_2 = -3 + 2i$

z and $\bar{z}$ are reflected across the real axis

Homework: Plot $z_1 = 2i$, $z_2 = 3$, $\bar{z}_1$ and $\bar{z}_2$ on an Argand diagram.

Example: Find the solutions of $z^2 - 6z + 10 = 0$. What is the relationship between the solutions? Plot the solutions on an Argand diagram.

$$z = \frac{6 \pm \sqrt{36 - 4(1)(10)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{-4}}{2}$$
$$= \frac{6 \pm \sqrt{4}\sqrt{-1}}{2}$$
$$= \frac{6 \pm 2i}{2}$$
$$= 3 \pm i$$



solutions are complex conjugates

Example: Let z and w be complex numbers. Prove the following properties of the complex conjugate.

(i) $z + \bar{z}$ is real
(ii) $z - \bar{z}$ is imaginary
(iii) $z\bar{z}$ is real
(iv) $\overline{z + w} = \bar{z} + \bar{w}$
(v) $\overline{zw} = \bar{z}\bar{w}$

Let $z = a + ib$
$w = c + id$

(i) $z + \bar{z} = (a+ib) + (a-ib) = 2a \leftarrow$ real

(ii) $z - \bar{z} = (a+ib) - (a-ib) = 2ib \leftarrow$ imag.

(iii) $z\bar{z} = (a+ib)(a-ib)$
$= a^2 - (ib)^2$ diff. of perfect squares
$= a^2 - i^2b^2 = a^2 + b^2 \leftarrow$ real.

(iv) $\overline{z+w} = \overline{(a+ib)+(c+id)}$

$= \overline{(a+c)+i(b+d)}$

$= (a+c)-i(b+d)$

$= (a-ib)+(c-id)$

$= \bar{z}+\bar{w}$

(v) LHS $= \overline{zw} = \overline{(a+ib)(c+id)}$

$= \overline{ac+iad+ibc+i^2bd}$

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$= \overline{(ac-bd)+i(ad+bc)}$

$= (ac-bd)-i(ad+bc)$

RHS $= \bar{z}\,\bar{w} = (a-ib)(c-id)$

$= ac-iad-ibc+i^2bd$

$= ac-iad-ibc-bd$

$= (ac-bd)-i(ad+bc)$

$=$ LHS

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Division of complex numbers

Suppose we want to divide two complex numbers:

$$\frac{a+ib}{c+id} \tag{2}$$

In order to get an answer in the form $x+iy$ we need to make the denominator real. To do so, we can make use of property (iii) of the previous example, which says that a complex number multiplied by its conjugate is real. So, we simply multiply top and bottom of (2) by the complex conjugate of the denominator, $c-id$.

$$\frac{a+ib}{c+id}\times\frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2-(id)^2}$$

$$= \frac{(a+ib)(c-id)}{c^2-i^2d^2}$$

$$= \frac{(a+ib)(c-id)}{c^2+d^2} \leftarrow \text{multiply out}$$

Let's try this on an example.

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Example: Express the following in cartesian form $x+iy$.

$$\frac{1+2i}{-1+3i} \times \frac{-1-3i}{-1-3i}$$

$$= \frac{(1+2i)(-1-3i)}{(-1)^2-(3i)^2}$$

$$= \frac{-1-3i-2i-6i^2}{1-9i^2}$$

$$= \frac{-1-5i+6}{1+9}$$

$$= \frac{5-5i}{10} = \frac{1}{2}-\frac{1}{2}i$$

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Example: Find $\text{Re}\left(\frac{1+5i}{2-2i}\right)$ and $\text{Im}\left(\frac{1+5i}{2-2i}\right)$.

$$\frac{1+5i}{2-2i} \times \frac{2+2i}{2+2i}$$

$$= \frac{(1+5i)(2+2i)}{2^2+2^2}$$

$$= \frac{2+2i+10i+10i^2}{8}$$

$$= \frac{2+12i-10}{8}$$

$$= \frac{-8+12i}{8} = -1+\frac{3}{2}i$$

$$\Rightarrow \text{Re}\left(\frac{1+5i}{2-2i}\right) = -1$$

$$\text{Im}\left(\frac{1+5i}{2-2i}\right) = \frac{3}{2}$$

Exercise: Let $z = 1-5i$ and $w = -2+i$. Express the following complex numbers in cartesian form $a+ib$ where a and b are real.

(i) w^2z (ii) $\dfrac{w}{w+2z}$

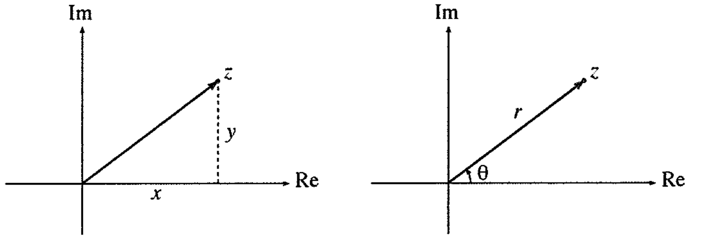
Answers: (i) $-17-19i$ (ii) $-\frac{1}{9}-\frac{2}{9}i$

2.3 Modulus and Argument

Consider a complex number z represented in the complex plane (Argand diagram).

The position of the complex number can be specified in two different ways:

- by its real and imaginary parts x and y, such that $z = x+iy$.
- by its distance r from the origin, and the angle θ around from the positive real axis.

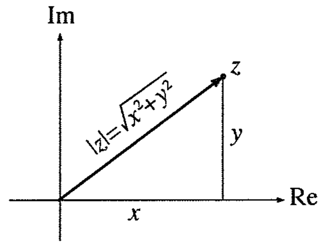


So the distance r from the origin and the angle θ give us an alternative way of specifying a complex number.

Definition: The **modulus** of z, denoted $|z|$, is the distance r of z from the origin in the complex plane.

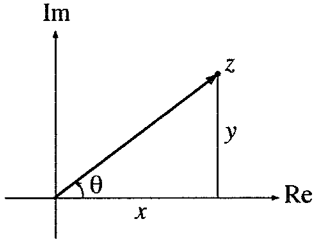
If $z = x+iy$ we can find $|z|$ by Pythagoras:

$$|z| = \sqrt{x^2+y^2}$$



Definition: The **argument** of z, denoted $\arg(z)$, is the angle θ that z makes with the positive real axis in the complex plane.

To determine the argument of a complex number $z = x + iy$, draw z in the complex plane and use standard triangles if possible to determine $\theta = \arg(z)$.



If θ is not a standard angle, we may note that

$$\tan(\theta) = \frac{y}{x}.$$

If θ is in the first or fourth quadrants we may then conclude $\theta = \arctan(\frac{y}{x})$ (recall that the range of arctan is $(\frac{\pi}{2}, \frac{\pi}{2})$).

Caution:

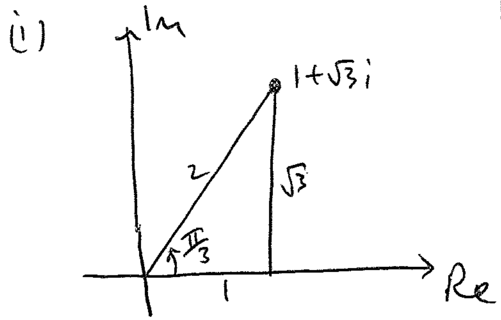
The argument of z is not unique, since adding multiples of 2π does not change the position of z in the complex plane.

However, there is only one value of the argument that satisfies $-\pi < \theta \leq \pi$. This is called the **principal argument** of z and is sometimes denoted $\text{Arg}(z)$ with a capital A.

Example: Find the modulus and argument of

(i) $1 + \sqrt{3}i$ (ii) $-3 - 3i$ (iii) $3 + 4i$ (iv) -7

(i)

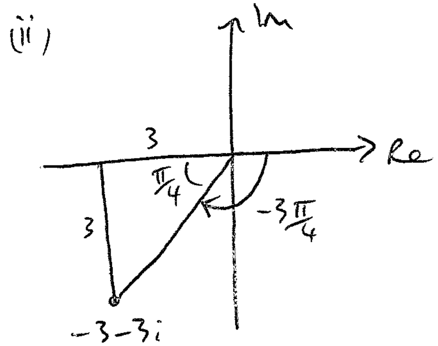


$|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= \sqrt{1+3} = 2$

$\arg(z) = \frac{\pi}{3}$

(ii)

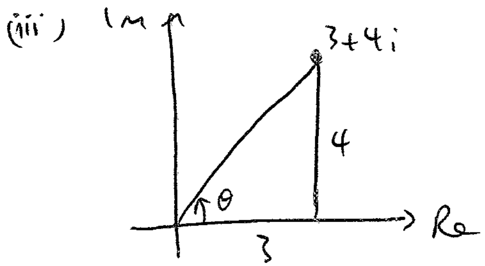


$|z| = \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{18} = 3\sqrt{2}$

$\arg(z) = -\frac{3\pi}{4}$

(iii)

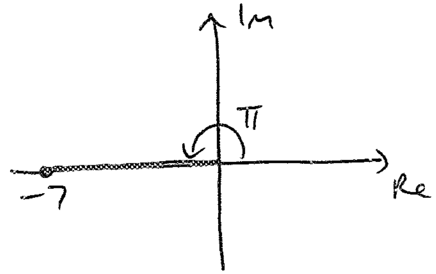


$|z| = \sqrt{3^2 + 4^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

$\arg(z) = ?$

$\tan\theta = \frac{4}{3} \Rightarrow \theta = \arctan(\frac{4}{3})$



$|z| = \sqrt{(-7)^2 + 0^2}$

$= 7$

$\arg(z) = \pi$

Homework: Find the modulus and argument of $z = 2 - 2i$.

Answer: $|z| = 2\sqrt{2}$, $\arg(z) = -\frac{\pi}{4}$

Properties of the Modulus and Argument

In Section 2.5 we will prove the following properties of the modulus and argument of complex numbers z and w.

1. $|zw| = |z||w|$
2. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
3. $\arg(zw) = \arg(z) + \arg(w)$
4. $\arg\left(\dfrac{z}{w}\right) = \arg(z) - \arg(w)$

Example: Using the above properties of the modulus, evaluate

$$\left|\frac{-2(3-i)(5+2i)}{(1+3i)(7-i)}\right|$$

$$= \frac{|-2|\,|3-i|\,|5+2i|}{|1+3i|\,|7-i|}$$ by properties 1 and 2

$$= \frac{2\sqrt{\cancel{3^2+(-1)^2}}\ \sqrt{5^2+2^2}}{\sqrt{\cancel{1^2+3^2}}\ \sqrt{7^2+(-1)^2}}$$

$$= \frac{2\sqrt{29}}{\sqrt{50}}$$

Example: Using the properties of the argument, evaluate

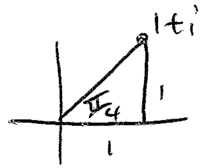
(i) $\arg\left((1+i)(-1+\sqrt{3}i)\right)$

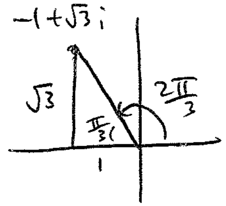
$$= \arg(1+i) + \arg(-1+\sqrt{3}i)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3}$$

$$= \frac{3\pi}{12} + \frac{8\pi}{12}$$

$$= \frac{11\pi}{12}$$





(ii) $\arg\left(\dfrac{-i}{-2+2i}\right)$

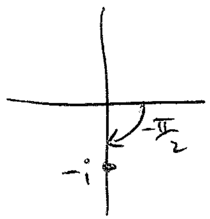
$$= \arg(-i) - \arg(-2+2i)$$

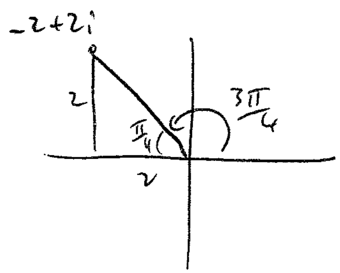
$$= \frac{-\pi}{2} - \frac{3\pi}{4}$$

$$= -\frac{2\pi}{4} - \frac{3\pi}{4}$$

$$= -\frac{5\pi}{4}$$

$$= \frac{3\pi}{4}$$

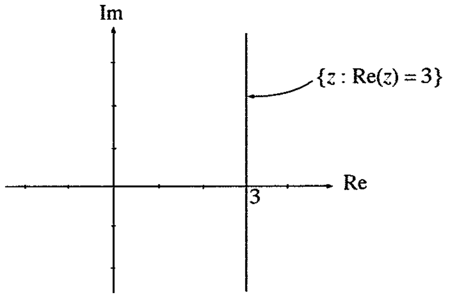




2.4 Sketching Regions in the Complex Plane

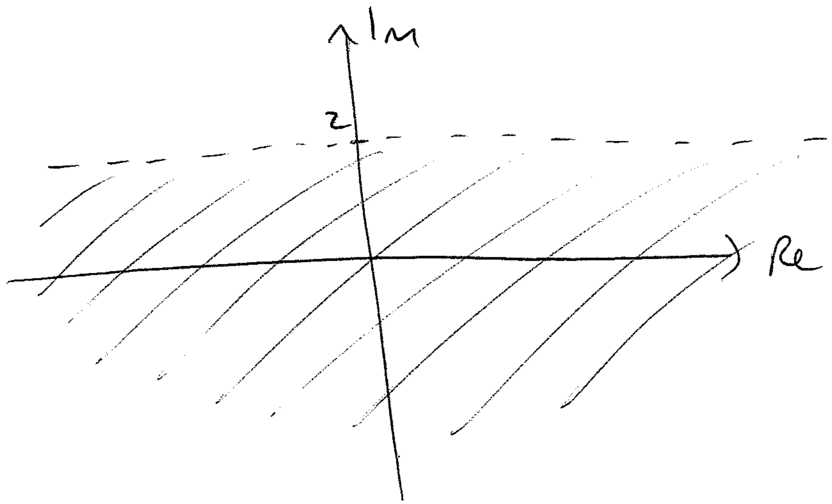
By considering the set of complex numbers that satisfy certain conditions, we obtain a corresponding region in the complex plane.

For example, $\{z : \text{Re}(z) = 3\}$, the set of all complex numbers whose real part is 3, can be represented in the complex plane by the vertical line intersecting the real axis at 3.



Example: Sketch the set of points in the complex plane satisfying

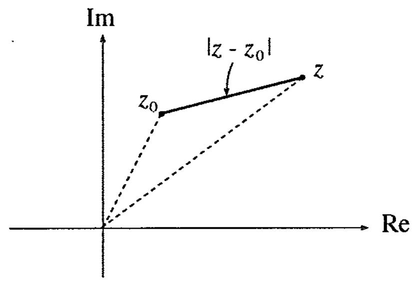
$$\text{Im}(z) < 2\,.$$



Example: Sketch the region in the complex plane given by

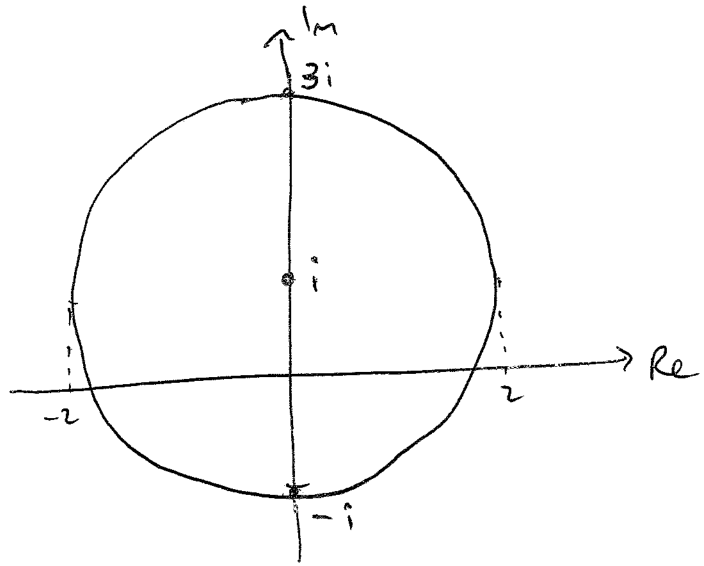
$$|z - i| = 2\,.$$

First, we recognise that if z and z_0 are complex numbers, then $|z - z_0|$ gives the distance between z and z_0 in the complex plane.



So in our example we want all complex numbers z such that the distance between z and i is 2.

This is represented by a circle centred at i, and of radius 2.



Example cont'd: Now find the cartesian equation of this curve.

We can find the cartesian equation (i.e. an equation in terms of x and y), by substituting $z = x + iy$ and simplifying.

$$|z - i| = 2$$

$$\Rightarrow \quad |x + iy - i| = 2$$

$$\Rightarrow \quad |x + i(y-1)| = 2$$

$$\Rightarrow \quad \sqrt{x^2 + (y-1)^2} = 2$$

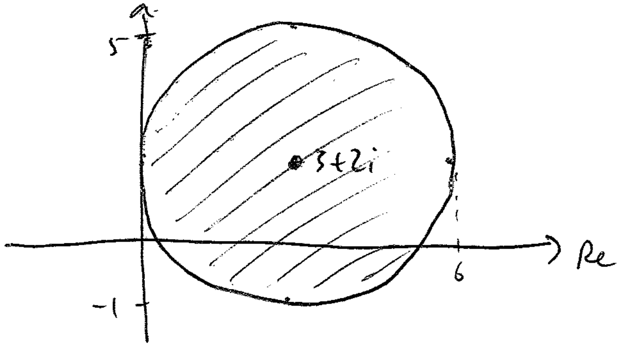
$$\Rightarrow \quad x^2 + (y-1)^2 = 4$$

So we have obtained the equation of the circle in cartesian form.

Example: Sketch the region given by $|z - 3 - 2i| \leq 3$.

$$|z - (3+2i)| \leq 3$$

"distance between z and $3+2i$ is ≤ 3"

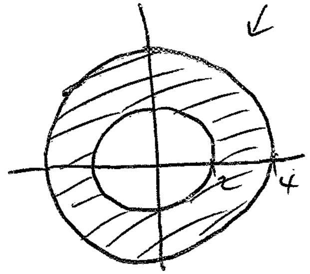


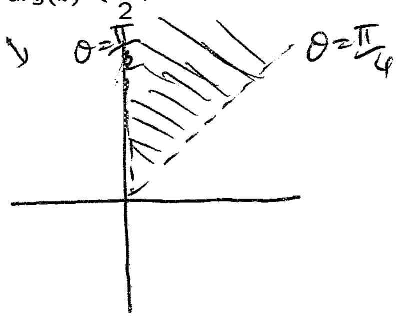
Homework: Find the cartesian equation of this region by substituting $z = x + iy$.

Answer: $(x-3)^2 + (y-2)^2 \leq 9$

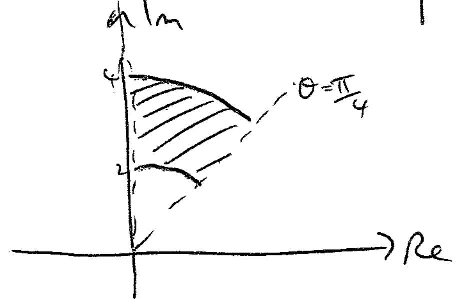
Example: Sketch the region given by

$$2 \leq |z| \leq 4 \quad \text{and} \quad \frac{\pi}{4} < \arg(z) < \frac{\pi}{2}.$$





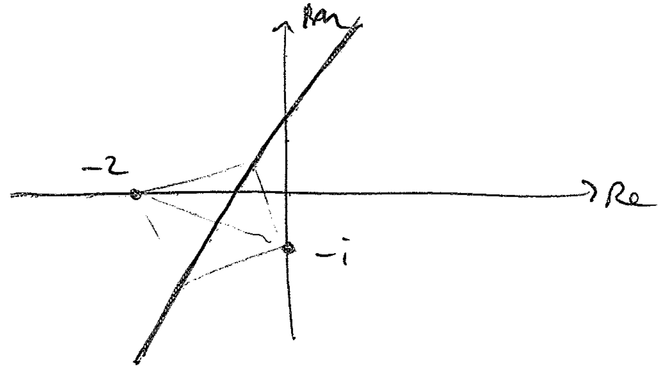
Intersection of these:



Example: Sketch the region given by $|z + 2| = |z + i|$.

$$|z - (-2)| = |z - (-i)|$$

"distance between z and -2 = distance between z and $-i$"



Homework: Find the cartesian equation of the above curve.

Answer: $y = 2x + \frac{3}{2}$

Example: Sketch the region given by

$$z = i\bar{z}\,.$$

$$\begin{aligned} x+iy &= i(\overline{x+iy}) \\ &= i(x-iy) \\ &= ix - i^2 y \\ &= ix + y \end{aligned}$$

Equate real & imaginary parts:

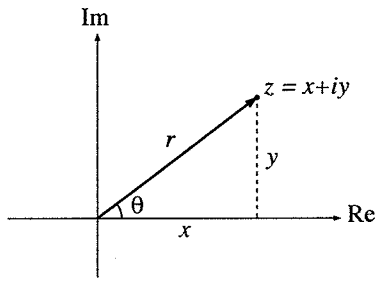
$$\left.\begin{aligned} \text{Re:}\quad & x=y \\ \text{Im:}\quad & y=x \end{aligned}\right\rangle \text{ same!} \quad \Rightarrow \quad y=x.$$

Homework: Sketch the region given by $z\bar{z} = 16$.

Answer: Circle of radius 4, centred at origin.

2.5 Polar Form

Recall that any complex number $z = x + iy$ can also be specified by its modulus r and argument θ.



From the diagram above, we see that

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

$$\Rightarrow \quad \boxed{x = r\cos\theta} \qquad \text{and} \qquad \boxed{y = r\sin\theta}$$

Substituting these back into z we obtain

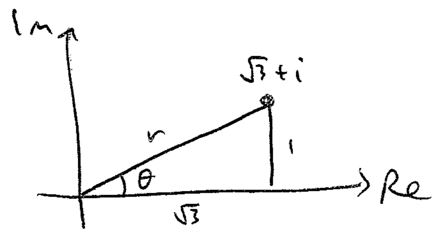
$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + i\,r\sin\theta \end{aligned}$$

And so we obtain the **polar form** of a complex number:

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

Example: Express the following complex numbers in polar form.

(i) $z = \sqrt{3} + i$



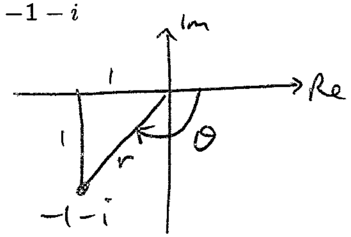
$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \frac{\pi}{6}$

$\Rightarrow$ polar form:

$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $z = -1 - i$



$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = -\frac{3\pi}{4}$

$z = \sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

Multiplication in polar form

Consider two complex numbers written in polar form:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

The product of these complex numbers is

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\big(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \underbrace{i^2}_{i^2=-1}\sin\theta_1\sin\theta_2\big)$$

$$= r_1 r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big)$$

Recall the compound angle formulas for cos and sin:

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

which shows that our expression for $z_1 z_2$ simplifies to

$$\boxed{z_1 z_2 = r_1 r_2\Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big)}$$

$$r(\cos\theta + i\sin\theta)$$

Interpreting this geometrically shows that the product of two complex numbers in polar form is obtained by multiplying their moduli $(r_1 r_2)$ and adding their arguments $(\theta_1+\theta_2)$:

$$\boxed{\begin{gathered}|z_1 z_2| = |z_1||z_2| \\ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)\end{gathered}}$$

$(r = r_1 r_2)$

$(\theta = \theta_1 + \theta_2)$

We already made use of these properties in Section 6.3.

Example: Describe geometrically what happens when a complex number z is multiplied by $w = i$.

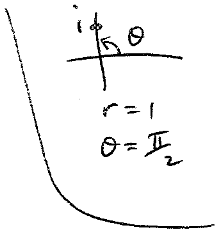
$z = r(\cos\theta + i\sin\theta)$ $\quad w = i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$zw = r\cdot 1\left(\cos\left(\theta+\frac{\pi}{2}\right) + i\sin\left(\theta+\frac{\pi}{2}\right)\right)$$

modulus stays same

argument increases by $\frac{\pi}{2}$

i.e. rotate anticlockwise by $\frac{\pi}{2}$



Homework: Describe geometrically what happens when a complex number is multiplied by $w = -i$.

Answer: It is rotated clockwise about the origin through an angle of $\frac{\pi}{2}$.

Division in polar form

Consider two complex numbers written in polar form:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Provided $z_2 \neq 0$, the quotient of these complex numbers is

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1\left(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \overbrace{i^2}^{i^2=-1}\sin\theta_1\sin\theta_2\right)}{r_2\,\underbrace{(\cos^2\theta_2 + \sin^2\theta_2)}_{=1}}$$

$$= \frac{r_1}{r_2}\Big((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\Big)$$

Recall the angle difference formulas for cos and sin:

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1 - \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$$

which shows that our expression for $\frac{z_1}{z_2}$ simplifies to

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big)}$$

Interpreting this geometrically shows that the product of two complex numbers in polar form is obtained by dividing their moduli $(\frac{r_1}{r_2})$ and subtracting their arguments $(\theta_1 - \theta_2)$:

$$\boxed{\begin{gathered}\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \\ \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\end{gathered}}$$

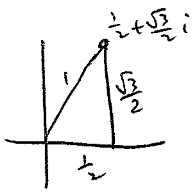
Example: Describe geometrically what happens when a complex number z is divided by $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

$z = r(\cos\theta + i\sin\theta)$ $\qquad w = 1(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$\Rightarrow \frac{z}{w} = \frac{r}{1}\left(\cos(\theta - \frac{\pi}{3}) + i\sin(\theta - \frac{\pi}{3})\right)$

modulus unchanged

argument decreases by $\frac{\pi}{3}$



$r = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$
$= 1$

$\theta = \frac{\pi}{3}$

$\Rightarrow$ rotation clockwise by $\frac{\pi}{3}$

2.6 The Complex Exponential

Recall that the polar form of a complex number z is

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$.

Definition: The **complex exponential** $e^{i\theta}$ is *defined* to be

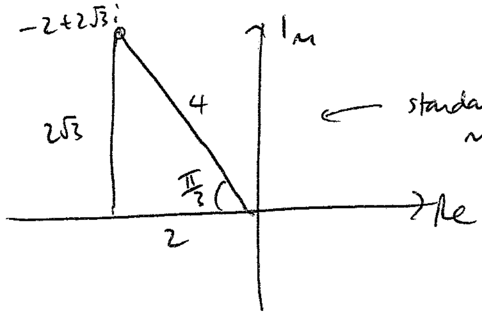
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

With this definition we can now write a complex number in **exponential polar form**

$$\boxed{z = re^{i\theta}}$$

where $r = |z|$ and $\theta = \arg(z)$.

Example: Express $z = -2 + 2\sqrt{3}i$ in exponential polar form $re^{i\theta}$.



standard triangle multiplied by 2

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$
$= \sqrt{4+12}$
$= 4$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow \boxed{z = 4e^{i\frac{2\pi}{3}}}$

Homework: Express $z = 1 - i$ in exponential polar form.

Answer: $z = \sqrt{2}e^{-i\frac{\pi}{4}}$

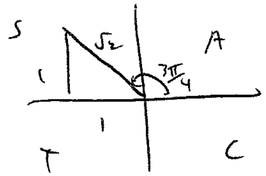
Example: Express $z = 5e^{i\frac{3\pi}{4}}$ in cartesian form $x + iy$.

$$z = 5e^{i3\pi/4}$$

$$= 5\left(\cos\left(\tfrac{3\pi}{4}\right) + i\sin\left(\tfrac{3\pi}{4}\right)\right)$$

$$= 5\left(-\tfrac{1}{\sqrt{2}} + i\,\tfrac{1}{\sqrt{2}}\right)$$

$$= -\frac{5}{\sqrt{2}} + i\,\frac{5}{\sqrt{2}}$$



Homework: Express $z = 4e^{-i\frac{\pi}{3}}$ in cartesian form.

Answer: $z = 2 - 2\sqrt{3}i$

Properties of the complex exponential

From the definition of the complex exponential and the properties of polar form that we have already shown, we obtain the following:

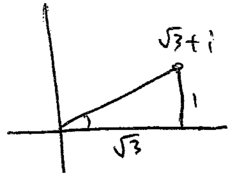
1. $e^{i0} = 1$
2. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
3. $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

Notice that these properties are consistent with the usual index laws for working with exponentials (and so are easy to remember!).

Because of this, and because of its compact form, we generally work with exponential polar form $re^{i\theta}$ instead of standard polar form $r(\cos\theta + i\sin\theta)$.

Example: If $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}i$, use exponential polar form to find $\frac{1}{z}$ and zw.

$z = \sqrt{3} + i$

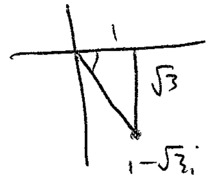


$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\theta = \frac{\pi}{6}$

$\Rightarrow \quad z = 2e^{i\pi/6}$

$w = 1 - \sqrt{3}i$



$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\theta = -\pi/3$

$w = 2e^{-i\pi/3}$

$\Rightarrow \quad \dfrac{1}{z} = \dfrac{1}{2e^{i\pi/6}}$

$= \frac{1}{2}e^{-i\pi/6}$

$zw = 2e^{i\pi/6} \cdot 2e^{-i\pi/3}$

$= 4e^{-i\pi/6}$

Exercise: express these in cartesian form using $e^{i\theta} = \cos\theta + i\sin\theta$

Ans: $\frac{\sqrt{3}}{4} - i\frac{1}{4}$, $2\sqrt{3} - 2i$

De Moivre's Theorem

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

De Moivre's theorem states that for any integer n:

$$z^n = r^n e^{in\theta}$$

$$\text{or} \qquad z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big)$$

Note that again, in exponential form, this result is consistent with the usual index laws

$$z^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^n e^{in\theta}$$

Idea of proof:

$z = re^{i\theta}$

$z^2 = z \cdot z = re^{i\theta} \cdot re^{i\theta} = r^2 e^{i(\theta+\theta)} = r^2 e^{i2\theta}$

$z^3 = z \cdot z^2 = re^{i\theta} \cdot r^2 e^{i2\theta} = r^3 e^{i(\theta+2\theta)} = r^3 e^{i3\theta}$

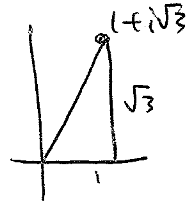
$z^4 = z \cdot z^3 = re^{i\theta} \cdot r^3 e^{i3\theta} = r^4 e^{i(\theta+3\theta)} = r^4 e^{i4\theta}$

By continuing the pattern we see the result for positive powers n.

Homework: See if you can deduce a similar pattern for negative powers n, using the properties of the complex exponential.

De Moivre's theorem can be used to avoid expanding the brackets when finding large powers of complex numbers.

Example: Use De Moivre's theorem to find $(1 + i\sqrt{3})^8$ in both exponential and cartesian form.



$r = 2$

$\theta = \frac{\pi}{3}$

$(1+i\sqrt{3})^8$

$= (2e^{i\frac{\pi}{3}})^8$

$= 2^8 e^{i\frac{8\pi}{3}}$ — exponential form

$= 2^8 e^{i\frac{6\pi}{3}} e^{i\frac{2\pi}{3}}$

$= 2^8 \cancel{e^{i2\pi}} e^{i\frac{2\pi}{3}}$

$e^{i2\pi} = e^{i0} = 1$

$= 2^8 e^{i\frac{2\pi}{3}}$

$= 2^8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$= 2^8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -2^7 + \sqrt{3}\cdot 2^7 i$

$= -128 + 128\sqrt{3}i$ — cartesian form

Example: Find $\left(\frac{2}{1+i}\right)^{14}$ in exponential and cartesian form.

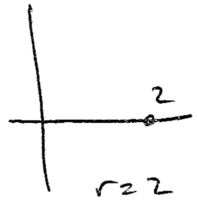
$$\left(\frac{2}{1+i}\right)^{14}$$

$$= \left(\frac{2e^{i0}}{\sqrt{2}e^{i\pi/4}}\right)^{14}$$

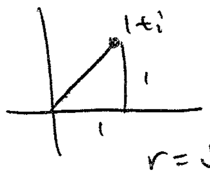
$$= \left(\sqrt{2}e^{-i\pi/4}\right)^{14}$$

$$= \sqrt{2}^{14}\, e^{-i\frac{14\pi}{4}}$$

$$= \sqrt{2}^{14}\, e^{-i\frac{16\pi}{4}}\, e^{i\frac{2\pi}{4}}$$



$r=2$
$\theta=0$



$r=\sqrt{2}$
$\theta=\frac{\pi}{4}$

$$= \sqrt{2}^{14}\, e^{-i4\pi}\, e^{i\frac{\pi}{2}}$$

$$= \sqrt{2}^{14}\, e^{i0}\, e^{i\frac{\pi}{2}}$$

$$= \sqrt{2}^{14}\, e^{i\frac{\pi}{2}}$$

$$= \sqrt{2}^{14}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \sqrt{2}^{14}\left(0 + i(1)\right)$$

$$= \sqrt{2}^{14}\, i$$

$$= 2^7\, i$$

$$= 128i$$

2.7 nth Roots of Complex Numbers

We have looked at how the complex exponential can be used to find integer powers of a complex number. Now we look at finding nth roots.

Suppose we wish to find the nth roots of a complex number w. So we want to find z such that

$$z = w^{\frac{1}{n}}$$

or

$$z^n = w$$

Using exponential polar form, let $z = re^{i\theta}$ and $w = se^{i\phi}$. So:

$$(re^{i\theta})^n = se^{i\phi}$$

$$\Rightarrow \quad r^n e^{in\theta} = se^{i\phi}$$

Equating the modulus and argument gives

$$r^n = s \qquad \text{and} \qquad e^{in\theta} = e^{i\phi}$$

$$\Rightarrow \quad r = s^{\frac{1}{n}} \qquad \Rightarrow \quad n\theta = \phi + 2k\pi \qquad k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \frac{1}{n}(\phi + 2k\pi) \qquad k \in \mathbb{Z}$$

So we have solved for $z = re^{i\theta}$, which gives the nth roots of w. Thus, the nth roots of $w = se^{i\phi}$ are:

$$\boxed{w^{\frac{1}{n}} = s^{\frac{1}{n}} e^{i(\frac{1}{n}(\phi+2k\pi))} \qquad \text{for } k = 0, 1, \ldots, n-1.}$$

get n different
nth roots

Note 1
Notice that $k = 0, 1, ..., n-1$ gives n different nth roots of w. We stop at $n-1$ since if $k = n$ we would be adding a whole multiple of 2π to the argument, which gives the same complex number as $k = 0$.

Note 2
You do **not** need to memorise the formula in the box. Instead, it is much easier to derive the nth roots of a complex number w by starting with $z^n = w$, expressing w in exponential polar form and taking nth roots to solve for z.

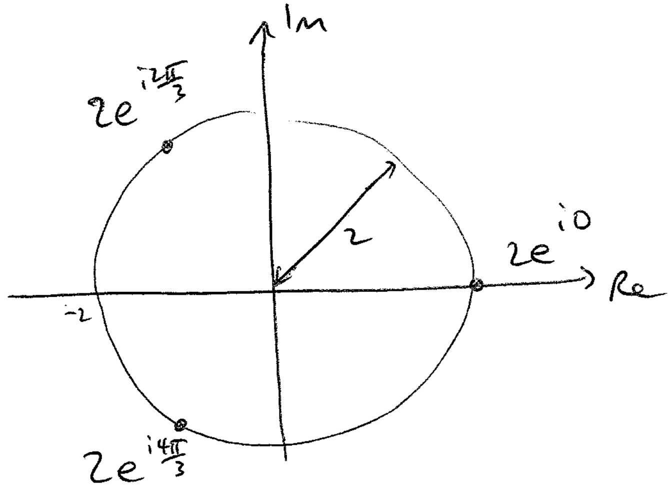
The following examples will illustrate the method.

Example: Find the cube roots of 8, sketch them in the complex plane, and express them in cartesian form.

Recall that over the reals, there is just one cube root of 8, namely 2. However, over the complex numbers we expect there to be three cube roots of 8!

$$z^3 = 8$$
$$= 8e^{i0}$$
$$= 8e^{i(0+2k\pi)}$$
$$z = 8^{\frac{1}{3}} e^{i\frac{1}{3}(0+2k\pi)} \qquad k = 0, 1, 2$$
$$= 2e^{i0}, \quad 2e^{i\frac{2\pi}{3}}, \quad 2e^{i\frac{4\pi}{3}}$$
$$(k=0) \qquad (k=1) \qquad (k=2)$$

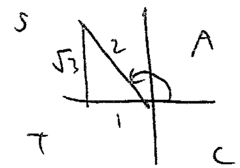
Sketch:



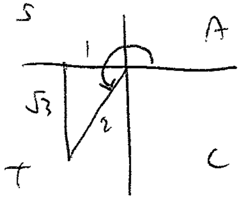
We see that the cube roots of 8 all have modulus 2, and are evenly spaced around a circle of radius 2 in the complex plane. This is a general property - the nth roots of any complex number are evenly spaced around the origin in the complex plane. Can you see why?

Cartesian form:

- $2e^{i0} = 2$
- $2e^{i\frac{2\pi}{3}} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= -1 + i\sqrt{3}$$



- $2e^{i\frac{4\pi}{3}} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$
$$= 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= -1 - i\sqrt{3}$$



$\Rightarrow$ roots are $2, -1 \pm i\sqrt{3}$

Example: Find the 6th roots of unity, and sketch these on the complex plane. You may keep your answers in exponential polar form.

Note: 'Unity' is just a fancy way of saying 1!

$$z^6 = 1$$
$$= 1e^{i0}$$
$$= 1\, e^{i(0+2k\pi)}$$

$$\Rightarrow \quad z = 1^{\frac{1}{6}}\, e^{i\frac{1}{6}(2k\pi)} \qquad k = 0, 1, \ldots, 5$$
$$= e^{i\frac{k\pi}{3}}$$
$$= \underset{=1}{e^{i0}},\ e^{i\frac{\pi}{3}},\ e^{i\frac{2\pi}{3}},\ \underset{=e^{i\pi}=-1}{e^{i\frac{3\pi}{3}}},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{5\pi}{3}}$$

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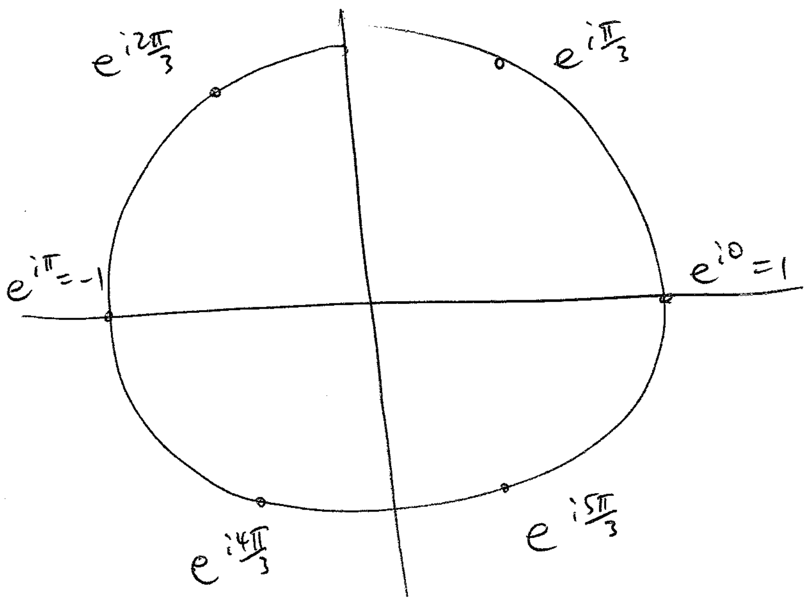

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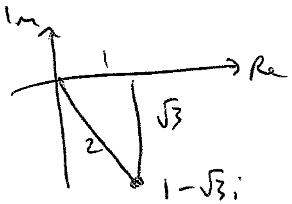
Example: Find the 4th roots of $1 - \sqrt{3}i$.



$r = 2$
$\theta = -\frac{\pi}{3}$

$$z^4 = 1 - \sqrt{3}i$$
$$= 2e^{-i\frac{\pi}{3}}$$
$$= 2e^{i(-\frac{\pi}{3} + 2k\pi)}$$

$$\Rightarrow \quad z = 2^{\frac{1}{4}}\, e^{i\frac{1}{4}(-\frac{\pi}{3} + 2k\pi)} \qquad k = 0, 1, 2, 3$$
$$= 2^{\frac{1}{4}}\, e^{i(-\frac{\pi}{12} + \frac{k\pi}{2})}$$
$$= 2^{\frac{1}{4}} e^{-i\frac{\pi}{12}},\ 2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \frac{\pi}{2})},$$
$$2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \pi)},\ 2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \frac{3\pi}{2})}$$

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$$= 2^{\frac{1}{4}} e^{-i\frac{\pi}{12}},\ 2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \frac{6\pi}{12})},$$
$$2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \frac{12\pi}{12})},\ 2^{\frac{1}{4}} e^{i(-\frac{\pi}{12} + \frac{18\pi}{12})}$$

$$= 2^{\frac{1}{4}} e^{-i\frac{\pi}{12}},\ 2^{\frac{1}{4}} e^{i\frac{5\pi}{12}},\ 2^{\frac{1}{4}} e^{i\frac{11\pi}{12}},$$
$$2^{\frac{1}{4}} e^{i\frac{17\pi}{12}}$$

Homework: Find the cube roots of $1 + i$

Answer: $2^{1/6}e^{i\pi/12}$, $2^{1/6}e^{i3\pi/4}$, $2^{1/6}e^{i17\pi/12}$

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2.8 Roots of polynomials

Having looked at nth roots of complex numbers, we now consider solving more general polynomial equations in a complex variable z.

Finding the roots of a polynomial $P(z)$ means solving $P(z) = 0$. One way to do this is to factorise $P(z)$ and set each of the factors equal to 0.

Recall that over the reals $\mathbb{R}$, some quadratics have roots because they can be factorised into linear factors, like

$$x^2 + 3x + 2 = (x + 2)(x + 1) ,$$

while others have no real roots because they cannot be factorised, such as

$$x^2 + 3x + 4 .$$

Over the complex numbers $\mathbb{C}$, however, we have the following beautiful result.

> **The Fundamental Theorem of Algebra**
> Every polynomial $P(z)$ of degree n can be factorised into n linear factors over $\mathbb{C}$, that is
> $$P(z) = a(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)$$
> where $a, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$.

This tells us that

every polynomial of degree n has exactly n roots over $\mathbb{C}$! (possibly repeated)

Another nice fact is the following:

> If the coefficients of $P(z)$ are real, then any non-real roots of $P(z)$ occur in complex conjugate pairs.

Example: Consider the polynomial $P(z) = z^3 - 3iz^2 - 2z$.

(i) How many roots do you expect this polynomial to have?

(ii) Factorise $P(z)$.

(iii) Hence find the roots of $P(z)$.

(iv) Verify your answers by substituting back into P and checking that $P(z) = 0$.

(i) expect 3 roots since cubic polynomial

(ii) $P(z) = z(\underbrace{z^2 - 3iz - 2})$

$$z = \frac{3i \pm \sqrt{(-3i)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{3i \pm \sqrt{9i^2 + 8}}{2}$$

$$= \frac{3i \pm \sqrt{-9+8}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm i}{2}$$

$$= \frac{4i}{2} \text{ or } \frac{2i}{2} = 2i \text{ or } i$$

$\Rightarrow$ $P(z) = z(z - 2i)(z - i)$

(iii) roots of $P(z)$ are $0, 2i, i$

(iv) $P(0) = 0^3 + 3i \cdot 0^2 - 2 \cdot 0 = 0$ ✓

$P(2i) = (2i)^3 - 3i(2i)^2 - 2(2i)$

$= 8i^3 - 12i^3 - 4i$ $\qquad i^3 = i^2 \cdot i = -i$

$= -8i + 12i - 4i = 0$ ✓

$P(i) = i^3 - 3i(i)^2 - 2i$

$= -i + 3i - 2i = 0$ ✓

Example: Solve $z^4 + z^2 - 12 = 0$.

If we let $w = z^2$ we get a quadratic:

$$w^2 + w - 12 = 0$$

$$\Rightarrow \quad (w+4)(w-3) = 0$$

$$\Rightarrow \quad w = -4 \quad \text{or} \quad w = 3$$

$$\Rightarrow \quad z^2 = -4 \qquad z^2 = 3$$

$$\Rightarrow \quad z = \pm 2i \qquad z = \pm\sqrt{3}$$

$\Rightarrow$ roots are $z = \pm 2i, \pm\sqrt{3}$

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Example: Solve $z^4 - 2z^2 + 4 = 0$, expressing your answers in exponential polar form.

Again let $w = z^2$

$$w^2 - 2w + 4 = 0$$

$$w = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2} \qquad \sqrt{-12} = \sqrt{4}\sqrt{3}\sqrt{-1}$$

$$= \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$$

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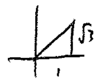
$$z^2 = 1 + \sqrt{3}i$$



$$= 2e^{i\frac{\pi}{3}}$$

$$= 2e^{i(\frac{\pi}{3} + 2k\pi)} \qquad k = 0, 1$$

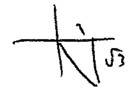
$$z = 2^{\frac{1}{2}} e^{i\frac{1}{2}(\frac{\pi}{3} + 2k\pi)}$$

$$= \sqrt{2}\, e^{i(\frac{\pi}{6} + k\pi)}$$

$$= \sqrt{2}\, e^{i\frac{\pi}{6}}, \sqrt{2}\, e^{i\frac{7\pi}{6}}$$

or

$$z^2 = 1 - \sqrt{3}i$$



$$= 2e^{-i\frac{\pi}{3}}$$

$$= 2e^{i(-\frac{\pi}{3} + 2k\pi)} \qquad k = 0, 1$$

$$z = 2^{\frac{1}{2}} e^{i\frac{1}{2}(-\frac{\pi}{3} + 2k\pi)}$$

$$= \sqrt{2}\, e^{i(-\frac{\pi}{6} + k\pi)}$$

$$= \sqrt{2}\, e^{-i\frac{\pi}{6}}, \sqrt{2}\, e^{i\frac{5\pi}{6}}$$

$$\Rightarrow z = \sqrt{2}e^{i\frac{\pi}{6}}, \sqrt{2}e^{-i\frac{\pi}{6}}, \sqrt{2}e^{i\frac{5\pi}{6}}, \sqrt{2}e^{i\frac{7\pi}{6}}$$

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Exercise: Express the above roots in cartesian form.

Answer: $\pm\frac{\sqrt{3}}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

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Example: Solve $P(z) = z^5 - z^4 + z^3 - z^2 + z - 1 = 0$.

HINT: Consider the product $(z+1)P(z)$.

$$(z+1)P(z) = (z+1)(z^5 - z^4 + z^3 - z^2 + z - 1)$$
$$= z^6 - \cancel{z^5} + \cancel{z^4} - \cancel{z^3} + \cancel{z^2} - \cancel{z}$$
$$+ \cancel{z^5} - \cancel{z^4} + \cancel{z^3} - \cancel{z^2} + \cancel{z} - 1$$
$$= z^6 - 1$$

So if we solve $z^6 - 1 = 0$

we are solving

$$(z+1)(\underbrace{z^5 - z^4 + z^3 - z^2 + z - 1}_{5 \text{ roots}}) = 0$$

$z = -1$

So the roots of $z^5 - z^4 + z^3 - z^2 + z - 1 = 0$

are the roots of $z^6 - 1 = 0$ except $z = -1$.

$\rightarrow$ Find roots of $z^6 - 1$:

$$z^6 - 1 = 0$$
$$\Rightarrow \quad z^6 = 1$$
$$= 1e^{i0}$$
$$= 1e^{i(0 + 2k\pi)}$$
$$\Rightarrow \quad z = 1^{\frac{1}{6}} e^{i\frac{2k\pi}{6}} \qquad k = 0, 1, \dots, 5$$
$$= e^{i\frac{k\pi}{3}}$$

$$= e^{i0},\ e^{i\frac{\pi}{3}},\ e^{i\frac{2\pi}{3}},\ e^{i\frac{3\pi}{3}},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{5\pi}{3}}$$

$e^{i0} \uparrow 1$, $e^{i\frac{3\pi}{3}} \uparrow -1$

$\rightarrow$ the roots of $P(z)$ are these roots except -1:

$$1,\ e^{i\frac{\pi}{3}},\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{5\pi}{3}}$$

Homework: Express the above roots in cartesian form.

Answer: $1, \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$