These notes are taken from the University of Melbourne course 620154 Calculus 1.
References refer to the subject textbook from 2008:
Anton, Bivens & Davis - *Calculus: Early Transcendentals*, 8th edition, Wiley, 2005.

Topic 1: Trigonometric Functions

In this topic we define the reciprocal trigonometric functions and inverse trigonometric functions, and derive several trigonometric identities.

1.1 Reciprocal trigonometric functions

1.2 Trigonometric formulae

1.3 Inverse trigonometric functions

1.1 Reciprocal trigonometric functions [Appendix A]

You will all be familiar with the three trigonometric functions *sine, cosine* and *tangent* and their graphs. Today we will define the reciprocals of these functions and sketch their graphs.

1.1.1 The cosecant function

The reciprocal of the sine function is called the *cosecant* function. It is abbreviated to "cosec" and is defined as:

$$\boxed{\operatorname{cosec}(x) = \frac{1}{\sin(x)} \qquad \text{provided } \sin(x) \neq 0.}$$

Since $\sin(x) = 0$ exactly when $x = n\pi$ for some $n \in \mathbb{Z}$, the domain of cosec is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.

Graph of $y = \operatorname{cosec}(x)$

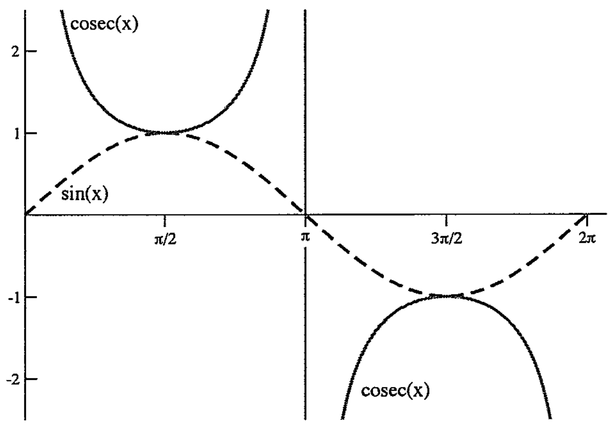
We derive the graph of $\operatorname{cosec}(x)$ from the graph of $\sin(x)$.

We already know that the **domain** of $\operatorname{cosec}(x)$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.

The range of sin is $[-1, 1]$ and since we are taking the reciprocal, the range of cosec is $\mathbb{R} \setminus (-1, 1)$.

The function sin has turning points at $\left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$ and therefore so does cosec.

The values of x for which cosec is not defined will be asymptotes.



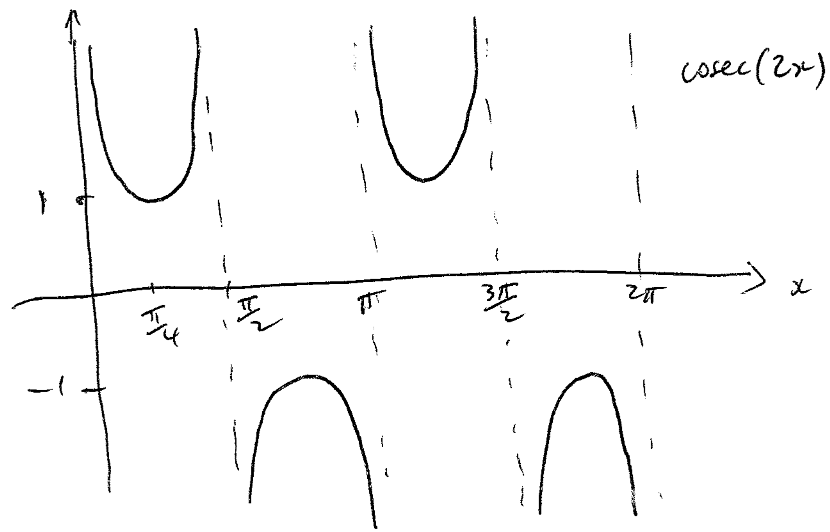
Example: Sketch the graph of cosec $(2x)$ over the domain $[0, 2\pi]$.

squashes graph towards y-axis by a factor of 2

→ period becomes π

→ • asymptotes at: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

• turning points at: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$



Homework: Sketch the graph of cosec $\left(x - \frac{\pi}{4}\right)$ over the domain $[0, 2\pi]$.

1.1.2 The secant function

The reciprocal of the cosine function is called the *secant* function. It is abbreviated to "sec" and is defined as:

$$\boxed{\sec(x) = \frac{1}{\cos(x)} \qquad \text{provided } \cos(x) \neq 0.}$$

Since $\cos(x) = 0$ exactly when $x = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$, the domain of cosec is $\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$.

Graph of $y = \sec(x)$

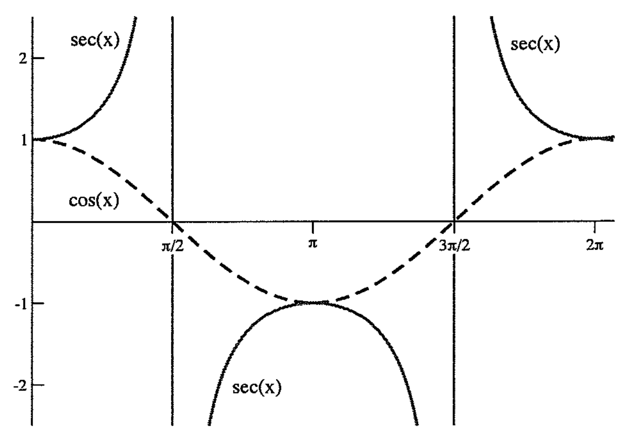
We derive the graph of $\sec(x)$ from the graph of $\cos(x)$.

The **domain** of $\sec(x)$ is $\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$.

The range of cos is $[-1, 1]$ and since we are taking the reciprocal, the range of sec is $\mathbb{R} \setminus (-1, 1)$.

The function cos has turning points at $\{n\pi : n \in \mathbb{Z}\}$ and therefore so does sec.

The values of x for which sec is not defined will be asymptotes.

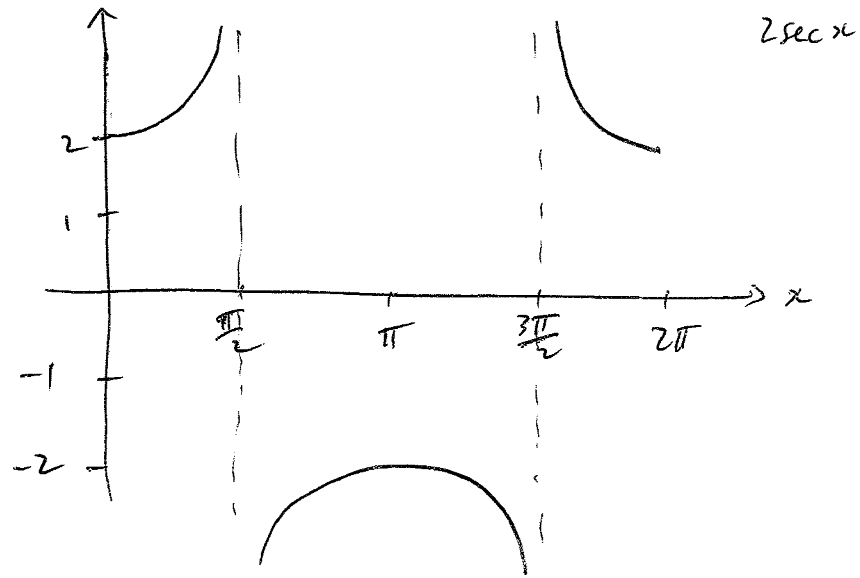


Example: Sketch the graph of $2\sec(x)$ over the domain $[0, 2\pi]$.

stretches vertically by a factor of 2

$\Rightarrow$ turning points will have y-values ± 2

- period unchanged



Homework: Sketch the graph of $-\sec(x)$ over the domain $[0, 2\pi]$.

1.1.3 The cotangent function

The reciprocal of the tangent function is called the *cotangent* function. It is abbreviated to "cot" and is defined as:

$$\boxed{\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \qquad \text{provided } \sin(x) \neq 0.}$$

Earlier we saw that $\sin(x) = 0$ exactly when $x = n\pi$ for some $n \in \mathbb{Z}$ so the domain of cot is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.

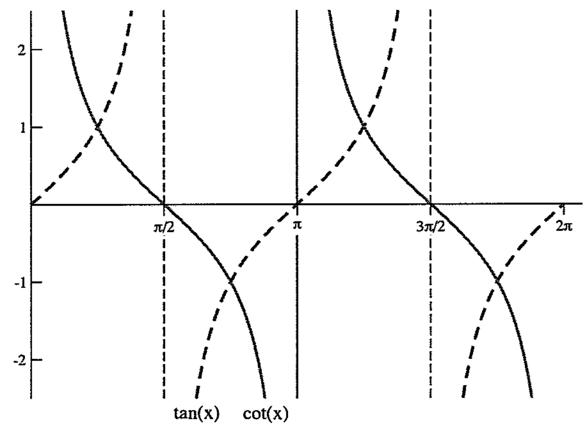
Graph of $y = \cot(x)$

We derive the graph of $\cot(x)$ from the graph of $\tan(x)$.

The **domain** of $\cot(x)$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.

The range of tan is $\mathbb{R}$ and since we are taking the reciprocal, the range of cot is $\mathbb{R}$.

The values of x for which tan is zero will be asymptotes for cot. Similarly, since tan is the reciprocal of cot, the zeros of cot occur where tan is undefined.



Example: Sketch the graph of $\cot\left(x + \frac{\pi}{4}\right)$ over the domain $[0, 2\pi]$.

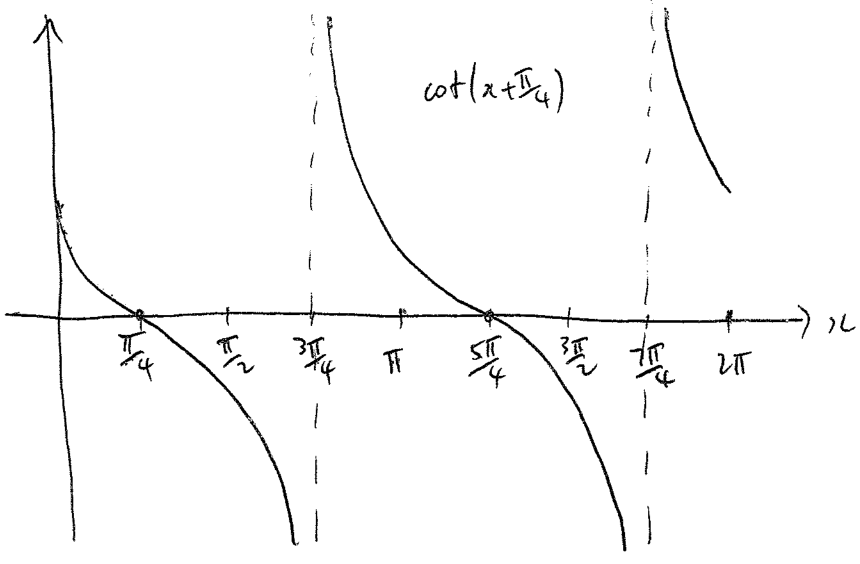
↓ shifted to left by $\frac{\pi}{4}$

$\Rightarrow$ asymptotes : $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

x - ints : $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$



Homework: Sketch the graph of $3\cot(x)$ over the domain $[0, 2\pi]$.

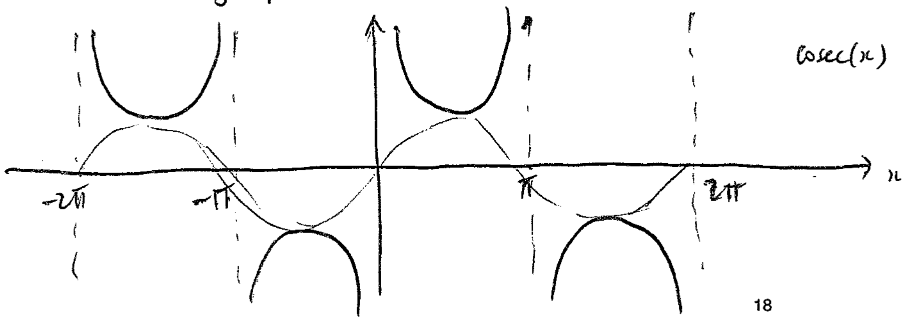
Harder example: Sketch the graph of $2\operatorname{cosec}\left(x-\frac{\pi}{4}\right)+1$ over the domain $[-2\pi, 2\pi]$.

multiply values by 2

shift to right by $\frac{\pi}{4}$

shift up by 1

Recall cosec graph over $[-2\pi, 2\pi]$:

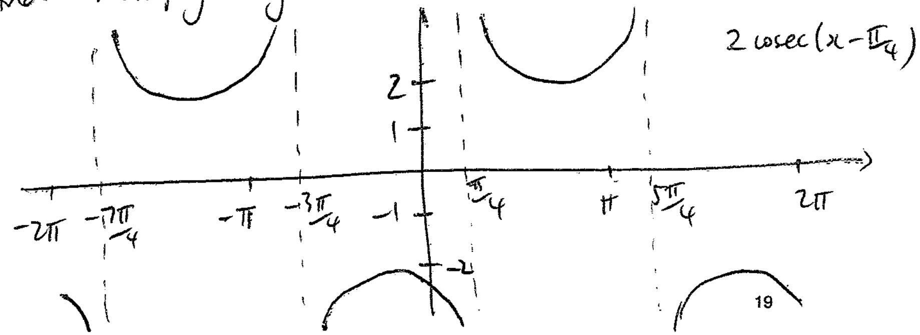


- shift to right by $\frac{\pi}{4}$:

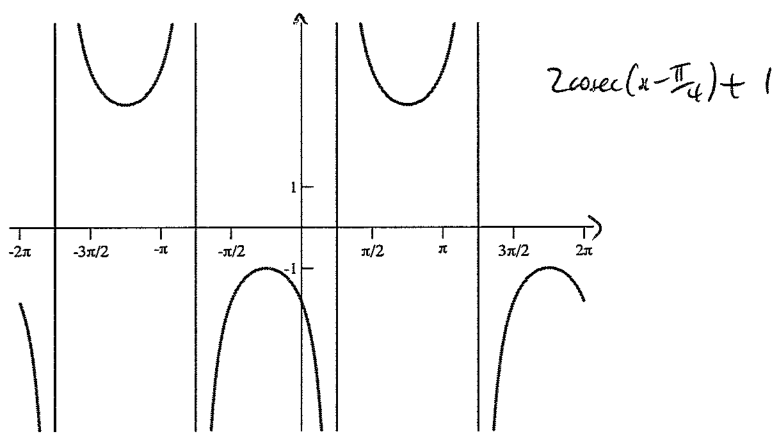
$\rightarrow$ asymptotes

$$-2\pi + \frac{\pi}{4} = -\frac{7\pi}{4}$$
$$-\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$
$$0 + \frac{\pi}{4} = \frac{\pi}{4}$$
$$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

- And multiply by 2:



- Finally, also shift up by 1:



Summary

$$\text{cosec}(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

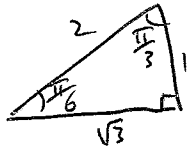
Example: Evaluate the following:

(a) $\text{cosec}(\frac{\pi}{6})$ (b) $\sec(\frac{\pi}{6})$ (c) $\cot(\frac{\pi}{6})$

(d) $\text{cosec}(\frac{3\pi}{4})$ (e) $\sec(\frac{3\pi}{4})$ (f) $\cot(\frac{3\pi}{4})$

(a) $\text{cosec}(\frac{\pi}{6}) = \frac{1}{\sin(\frac{\pi}{6})} = \frac{1}{\frac{1}{2}} = 2$

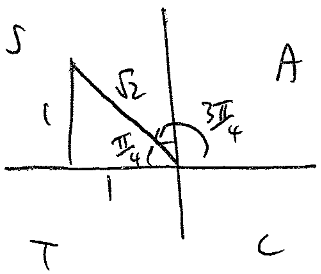
(b) $\sec(\frac{\pi}{6}) = \frac{1}{\cos(\frac{\pi}{6})} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$



(c) $\cot(\frac{\pi}{6}) = \frac{1}{\tan(\frac{\pi}{6})} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$

(d) $\text{cosec}(\frac{3\pi}{4}) = \frac{1}{\sin(\frac{3\pi}{4})}$

$= \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$



(e) $\sec(\frac{3\pi}{4}) = \frac{1}{\cos(\frac{3\pi}{4})} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$

(f) $\cot(\frac{3\pi}{4}) = \frac{1}{\tan(\frac{3\pi}{4})} = \frac{1}{-1} = -1$

Additional questions

You can now attempt a selection of exercises from 5-12, 15-16, 20-27 in Appendix A in the textbook.

1.2 Trigonometric Formulae [Appendix A]

An identity is an expression written in terms of a variable, which is true for all values of the variable in its implied domain. A familiar example of a trigonometric identity is:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

This is called the *Pythagorean identity*. It is true for all values of $\theta \in \mathbb{R}$. We will now look at some other trigonometric formulae.

In particular, we will look at:

- Trigonometric identities
- Compound and Double Angle formulae

1.2.1 Trigonometric identities

We start with the Pythagorean identity

$$\sin^2(\theta) + \cos^2(\theta) = 1. \qquad (1)$$

From this identity, we can derive identities involving reciprocal trigonometric functions. If we divide both sides of equation (1) by $\cos^2(\theta)$, we obtain:

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 = \frac{1}{\cos^2(\theta)} \qquad \text{provided } \cos(\theta) \neq 0$$

which is the same as

$$\boxed{\tan^2(\theta) + 1 = \sec^2(\theta) \qquad \text{for } \cos(\theta) \neq 0}$$

We can also divide equation (1) by $\sin^2(\theta)$ to obtain:

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \qquad \text{provided } \sin(\theta) \neq 0$$

which is the same as

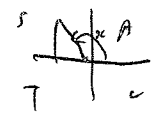
$$\boxed{1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta) \qquad \text{for } \sin(\theta) \neq 0}$$

These identities can be used to evaluate or to simplify trigonometric expressions.

Example: Simplify the expression $\sin^2(x)(1 + \cot^2(x))$.

$$\sin^2 x \, (1 + \cot^2 x)$$
$$= \sin^2 x \operatorname{cosec}^2 x$$
$$= \sin^2 x \cdot \frac{1}{\sin^2 x}$$
$$= 1$$

Example: If $x \in \left[\frac{\pi}{2}, \pi\right]$, and $\sin(x) = \frac{4}{5}$, find:



(a) $\cos(x)$ (b) $\cot(x)$.

(a) $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \left(\frac{4}{5}\right)^2 + \cos^2 x = 1$

$\Rightarrow \cos^2 x = 1 - \frac{16}{25}$

$= \frac{9}{25}$

$\Rightarrow \cos x = -\sqrt{\frac{9}{25}}$

$= -\frac{3}{5}$

cos -ve in 2nd quadrant

(b) $\cot x = \frac{1}{\tan x}$

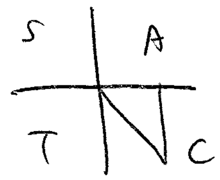
$= \frac{\cos x}{\sin x}$

$= \frac{-3/5}{4/5}$

$= -\frac{3}{4}$

Example: If $x \in \left[\frac{3\pi}{2}, 2\pi\right]$, and $\sec(x) = \frac{3}{2}$, find the exact values of:

(a) $\tan(x)$ (b) $\text{cosec}(x)$.



(a) Need identity relating tan and sec.

$$\tan^2 x + 1 = \sec^2 x$$
$$= \left(\tfrac{3}{2}\right)^2$$
$$\Rightarrow \quad \tan^2 x = \tfrac{9}{4} - 1$$
$$= \tfrac{5}{4}$$
$$\Rightarrow \quad \tan x = -\sqrt{\tfrac{5}{4}} = -\frac{\sqrt{5}}{2}$$

tan $-$ve in 4^{th} quadrant

(b) $\text{cosec}\, x = \dfrac{1}{\sin x}$ but don't know $\sin x$.

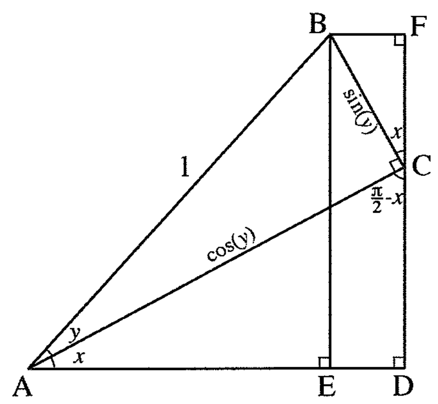
$\rightarrow$ look for identity with $\text{cosec}\, x$

$$1 + \cot^2 x = \text{cosec}^2 x$$
$$1 + \frac{1}{\tan^2 x} = \text{cosec}^2 x$$
$$1 + \frac{1}{5/4} = \text{cosec}^2 x$$
$$1 + \tfrac{4}{5} = \text{cosec}^2 x$$
$$\tfrac{9}{5} = \text{cosec}^2 x$$
$$\text{cosec}\, x = -\sqrt{\tfrac{9}{5}} = -\frac{3}{\sqrt{5}}$$

sin $-$ve in 4^{th} quadrant, so cosec $-$ve too

1.2.2 Compound angle formulae

Consider the right angled triangles pictured below.



Let the length of AB be equal to 1, the angle $DAC = x$ and the angle $BAC = y$. Then $AC = \cos(y)$ and $BC = \sin(y)$. The angle FCB is also equal to x, as shown.

So, in triangle ADC we have:

$$\sin(x) = \frac{opp}{hyp}$$
$$= \frac{DC}{AC}$$
$$= \frac{DC}{\cos(y)}$$

so

$$DC = \sin(x)\cos(y) \,.$$

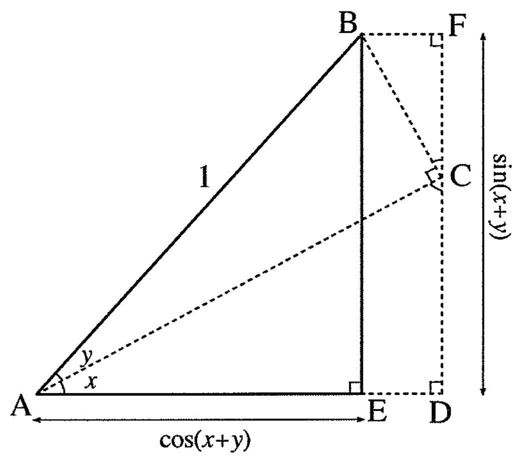
Similarly, by considering $\cos(x)$ in triangle ADC we find:

$$AD = \cos(x)\cos(y) \,.$$

And then, from triangle BCF,

$$BF = \sin(x)\sin(y)$$
$$CF = \cos(x)\sin(y)$$

We would like to find expressions for sin and cos of the *compound angle* $x+y$, as shown on the diagram below.



From the diagram,

$$\begin{aligned}\sin(x+y) &= DF\\ &= DC + CF\\ &= \sin(x)\cos(y) + \cos(x)\sin(y).\end{aligned}$$

Therefore

$$\boxed{\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)}$$

Similarly,

$$\begin{aligned}\cos(x+y) &= AE\\ &= AD - DE\\ &= AD - BF \qquad (\text{since } BF = DE)\\ &= \cos(x)\cos(y) - \sin(x)\sin(y).\end{aligned}$$

Therefore

$$\boxed{\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)}$$

By replacing the angle y with $-y$, and recalling that

$$\cos(-y) = \cos(y) \qquad \text{and} \qquad \sin(-y) = -\sin(y)$$

we can also deduce:

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y))\\ &= \sin(x)\cos(-y) + \cos(x)\sin(-y)\\ &= \sin(x)\cos(y) - \cos(x)\sin(y)\end{aligned}$$

$$\boxed{\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)}$$

and

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y))\\ &= \cos(x)\cos(-y) - \sin(x)\sin(-y)\\ &= \cos(x)\cos(y) + \sin(x)\sin(y)\end{aligned}$$

$$\boxed{\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)}$$

We can use these identities to deduce compound angle formulae for the tangent function:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

Dividing the numerator and denominator by $\cos(x)\cos(y)$, this expression simplifies to:

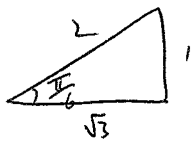
$$\boxed{\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}}$$

And similarly,

$$\boxed{\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}}$$

Example: Simplify the expression $\cos\left(x + \frac{\pi}{6}\right)$.

$$\cos\left(x+\tfrac{\pi}{6}\right) = \cos x \cos\tfrac{\pi}{6} - \sin x \sin\tfrac{\pi}{6}$$
$$= \cos x \cdot \frac{\sqrt{3}}{2} - \sin x \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$



Example: Simplify the expression $\operatorname{cosec}\left(x - \frac{\pi}{2}\right)$.

$$\operatorname{cosec}\left(x-\tfrac{\pi}{2}\right) = \frac{1}{\sin\left(x-\frac{\pi}{2}\right)}$$
$$= \frac{1}{\sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}}$$
$$= \frac{1}{\sin x \cdot 0 - \cos x \cdot 1}$$
$$= \frac{1}{-\cos x}$$
$$= -\sec x$$

Example: Find the exact value of $\tan\left(\frac{7\pi}{12}\right)$.

$$\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$$
$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$
$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$$
$$= \frac{1+\sqrt{3}}{1-1.\sqrt{3}}$$
$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{1+2\sqrt{3}+3}{1-3} = \frac{4+2\sqrt{3}}{-2}$$
$$= -2-\sqrt{3}$$

Homework: Find the exact value of $\cos(\frac{5\pi}{12})$.

Answer: $\frac{\sqrt{3}-1}{2\sqrt{2}}$

1.2.3 Double angle formulae

We now deduce the corresponding *double angle formulae*, for sin, cos and tan of a double angle, $2x$. These can be found by starting with the compound angle formulae.

Recall for sin we have:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

If we let $y = x$ then we have:

$$\begin{aligned}\sin(2x) &= \sin(x+x)\\ &= \sin(x)\cos(x) + \cos(x)\sin(x)\\ &= 2\sin(x)\cos(x).\end{aligned}$$

So the double angle formula for the sine function is:

$$\boxed{\sin(2x) = 2\sin(x)\cos(x)}$$

Similarly for cos we have:

$$\begin{aligned}\cos(2x) &= \cos(x+x)\\ &= \cos(x)\cos(x) - \sin(x)\sin(x)\\ &= \cos^2(x) - \sin^2(x).\end{aligned}$$

So:

$$\boxed{\cos(2x) = \cos^2(x) - \sin^2(x)}$$

We can obtain two alternative versions of this formula by recalling that $\sin^2(x) + \cos^2(x) = 1$, so:

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= (1 - \sin^2(x)) - \sin^2(x)\\ &= 1 - 2\sin^2(x)\\ &= 1 - 2(1 - \cos^2(x))\\ &= 2\cos^2(x) - 1.\end{aligned}$$

And finally using the compound angle formula for $\tan(x+y)$ we have:

$$\begin{aligned}\tan(2x) &= \tan(x+x)\\ &= \frac{\tan(x)+\tan(x)}{1-\tan(x)\tan(x)}\\ &= \frac{2\tan(x)}{1-\tan^2(x)}.\end{aligned}$$

So:

$$\boxed{\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}}$$

Homework: Simplify the expression $\dfrac{\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$.

Answer: $\frac{1}{2}\tan(x)$

1.2.4 Summary

Compound Angle Formulae

1. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
2. $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
3. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
4. $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
5. $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
6. $\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

Double Angle Formulae

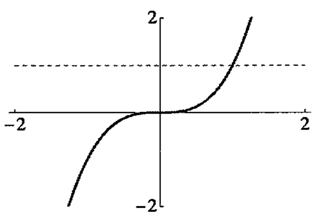
1. $\sin(2x) = 2\sin(x)\cos(x)$
2. $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
3. $\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

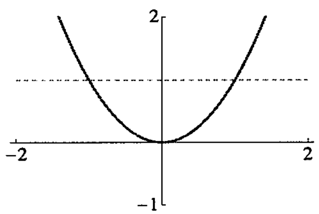
Additional questions

You can now attempt a selection of exercises from Appendix A exercises 45-58 in the textbook.

1.3 Inverse trigonometric functions [Chapter 1.5]

Recall that a function f is **one-to-one** if for each element y in the codomain of f there is *at most one* x in the domain of f such that $f(x) = y$. An easy way to see if a function is one-to-one or not, is to draw a horizontal line through the graph of f.





The function is one-to-one if there is no horizontal line which cuts the graph in more than one place. Here the graph on the left is one-to-one, but the graph on the right is not.

Clearly, the functions sin, cos and tan are not one-to-one because they are periodic. However we can restrict the domains of these functions to obtain one-to-one functions. For these restricted functions we can then define inverse functions.

1.3.1 The sine and inverse sine functions

We are going to define an inverse for the sine function.

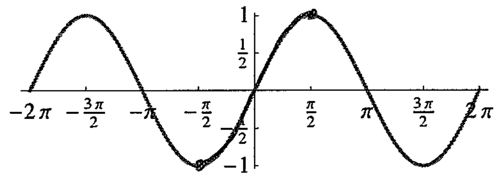


Figure 1. Graph of the sine function on $[-2\pi, 2\pi]$

To do so, we need to restrict the domain of the sine function so that the new function is one-to-one on this restricted domain, but the range of the function is not restricted.

Clearly many choices are possible. To avoid confusion, it is widely accepted that the restricted domain should be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

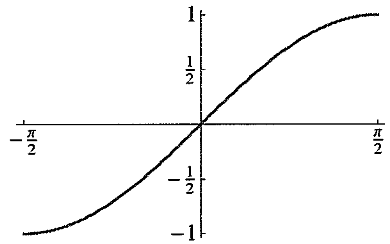
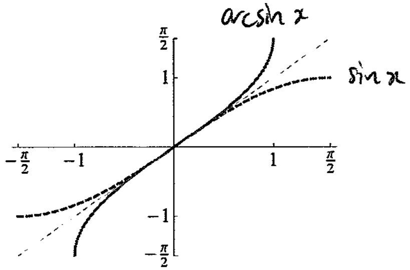


Figure 2. Graph of the sine function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Notice that the range of this restricted function is still $[-1, 1]$.
Now we can obtain an inverse function, by reflecting the graph through the line $y = x$.



We call this function *arcsine* (denoted arcsin).
The domain of this function is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

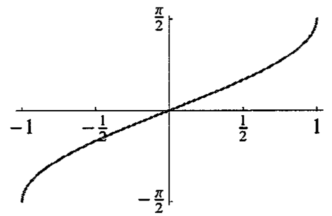


Figure 3. Graph of the arcsine function

This function is also referred to in some texts as Sin^{-1} where the capital S denotes the restricted domain and the index -1 means inverse rather than reciprocal.

In this subject we will only use the arcsin **notation.**

This avoids potential confusion between $\mathrm{Sin}^{-1}(x)$ and $\frac{1}{\sin(x)}$.

$$\arcsin x \neq \frac{1}{\sin x} = \operatorname{cosec} x$$

The function $\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies:

$$\boxed{\theta = \arcsin(x) \quad \Rightarrow \quad \sin(\theta) = x}$$

Note that the reverse implication holds only if θ is in the range of arcsin, i.e. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using properties of inverse functions, we can also say that for any $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\arcsin(\sin(x)) = x$$

and for any $y \in [-1, 1]$,

$$\sin(\arcsin(y)) = y.$$

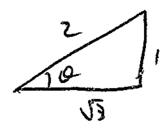
Example: Evaluate $\arcsin\left(\frac{1}{2}\right)$.

$\theta = \arcsin\left(\frac{1}{2}\right)$

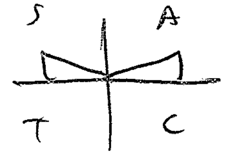
$\Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \cancel{\frac{5\pi}{6}}$

↑ in range of arcsin



$\theta = \frac{\pi}{6}$



Example: Simplify $\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right)$.

$\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

since $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ = range of arcsin

1.3.2 The cosine and inverse cosine functions

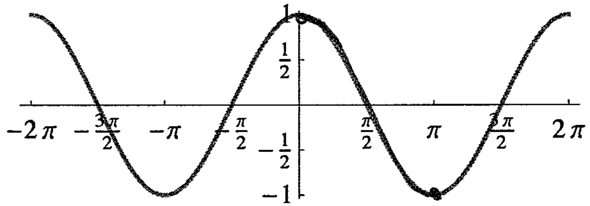


Figure 4. Graph of the cosine function on $[-2\pi, 2\pi]$

In a similar way we can restrict the domain of the cosine function to obtain a new one-to-one function.

In this case the restricted domain is chosen to be $[0, \pi]$.

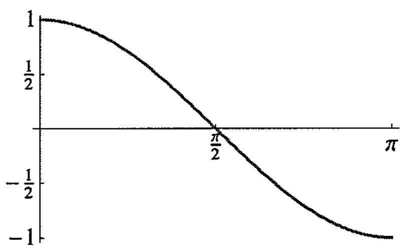
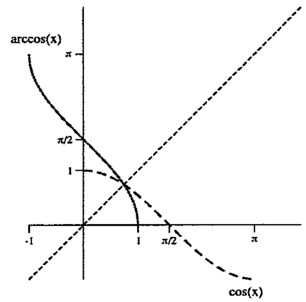


Figure 5. Graph of the cosine function on $[0, \pi]$

The range of this restricted function is still $[-1, 1]$.

We can obtain an inverse function, by reflecting the graph through the line $y = x$.



We call this function *arccosine* (denoted arccos).
The domain of this function is $[-1, 1]$ and the range is $[0, \pi]$.

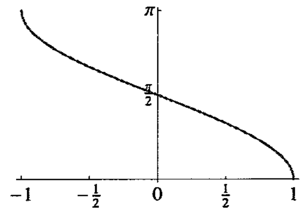


Figure 6. Graph of the arccosine function

The function $\arccos : [-1, 1] \to [0, \pi]$ satisfies:

$$\boxed{\theta = \arccos(x) \qquad \Rightarrow \qquad \cos(\theta) = x}$$

Note that the reverse implication holds only if θ is in the range of arccos, i.e. $\theta \in [0, \pi]$.

Using properties of inverse functions, we can also say that for any $x \in [0, \pi]$,

$$\arccos(\cos(x)) = x$$

and for any $y \in [-1, 1]$,

$$\cos(\arccos(y)) = y.$$

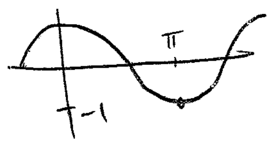
Example: Evaluate $\arccos(-1)$.

Let $\theta = \arccos(-1)$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pi$

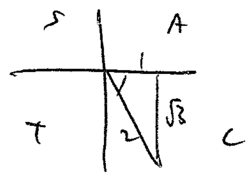
↑ in range of arccos, $[0, \pi]$



Example: Evaluate $\arccos\left(\sin\left(-\frac{\pi}{3}\right)\right)$.

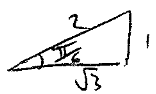
$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

↖ 4th quad., sin -ve.



let $\theta = \arccos\left(\sin\left(-\frac{\pi}{3}\right)\right)$

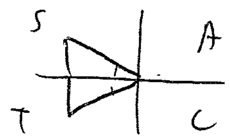
$= \arccos\left(-\frac{\sqrt{3}}{2}\right)$



$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = \frac{5\pi}{6}, \cancel{\frac{7\pi}{6}}$

↑ in range of arccos, $[0, \pi]$.



1.3.3 The tangent and inverse tangent functions

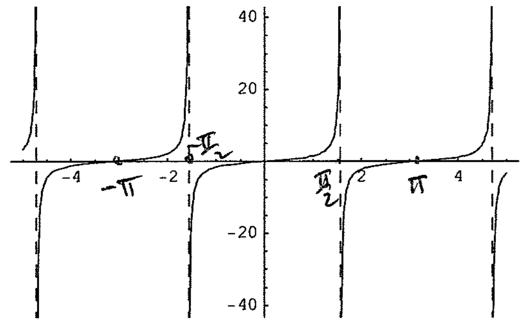


Figure 7. Graph of the tangent function on $[-2\pi, 2\pi]$

In this case we can restrict the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain a new one-to-one function.

Notice that the endpoints are not included in this case. Why?

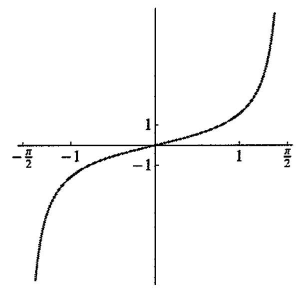
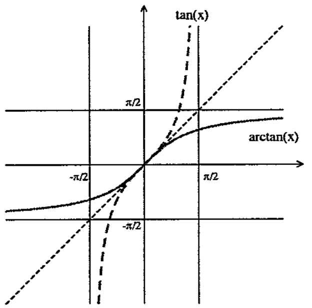


Figure 8. Graph of the tangent function on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The range of this restricted function is still $\mathbb{R}$.

We can obtain an inverse function, by reflecting the graph through the line $y = x$.



We call this function *arctangent* (denoted arctan).
The domain of this function is $\mathbb{R}$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

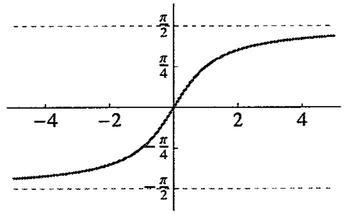


Figure 9. Graph of the arctangent function

The function $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies:

$$\boxed{\theta = \arctan(x) \quad \Rightarrow \quad \tan(\theta) = x}$$

Note that the reverse implication holds only if θ is in the range of arctan, i.e. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Using properties of inverse functions, we can also say that for any $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\arctan(\tan(x)) = x$$

and for any $y \in \mathbb{R}$,

$$\tan(\arctan(y)) = y.$$

Beware! $\arctan \neq \frac{\arcsin}{\arccos}$

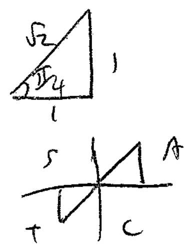
Example: Evaluate $\arctan(1)$.

Let $\theta = \arctan(1)$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \pi/4, \cancel{-3\pi/4}$

in range of arctan, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



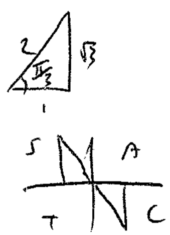
Example: Evaluate $\sin(\arctan(-\sqrt{3}))$.

Let $\theta = \arctan(-\sqrt{3})$

$\Rightarrow \tan\theta = -\sqrt{3}$

$\Rightarrow \theta = \cancel{2\pi/3}, -\pi/3$

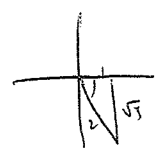
in range of arctan, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\Rightarrow \sin(\arctan(-\sqrt{3})) = \sin(-\pi/3)$

$= -\frac{\sqrt{3}}{2}$

sin -ve in 4^{th} quadrant

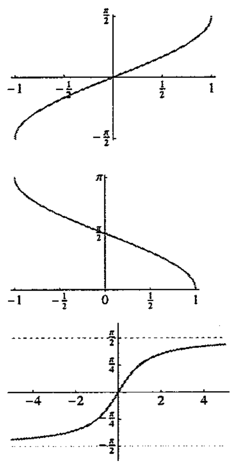


Summary

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\arccos : [-1, 1] \to [0, \pi]$$

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Homework: Evaluate the following:

(a) $\cos\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)\right)$

(b) $\arccos\left(\tan\left(-\frac{\pi}{4}\right)\right)$

(c) $\arctan\left(\sin\left(\pi\right)\right)$

(d) $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Answers: (a) $\frac{1}{\sqrt{2}}$ (b) π (c) 0 (d) $\frac{\pi}{3}$

Additional questions

You can attempt a selection of problems from 35-43 in Chapter 1.5 of the textbook.